Preface

Here are the solutions to the practice problems for my Calculus II notes. Some solutions will have more or less detail than other solutions. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you’ve reached the level of working the harder problems then you will probably already understand the basics fairly well and won’t need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven’t been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.
**Directional Derivatives**

1. Determine the gradient of the following function.

   \[ f(x, y) = x^2 \sec(3x) - \frac{x^2}{y^3} \]

   **Solution**
   Not really a lot to do for this problem. Here is the gradient.

   \[
   \nabla f = \left\langle f_x, f_y \right\rangle = \left\langle 2x \sec(3x) + 3x^2 \sec(3x) \tan(3x) - \frac{2x}{y^3}, \frac{3x^2}{y^4} \right\rangle
   \]

2. Determine the gradient of the following function.

   \[ f(x, y, z) = x \cos(xy) + z^2 y^4 - 7xz \]

   **Solution**
   Not really a lot to do for this problem. Here is the gradient.

   \[
   \nabla f = \left\langle f_x, f_y, f_z \right\rangle = \left\langle \cos(xy) - xy \sin(xy) - 7z, -x^2 \sin(xy) + 4z^2 y^3, 2zy^4 - 7x \right\rangle
   \]

3. Determine \( D_\vec{v} f \) for \( f(x, y) = \cos\left(\frac{x}{y}\right) \) in the direction of \( \vec{v} = \langle 3, -4 \rangle \).

   **Step 1**
   Okay, we know we need the gradient so let’s get that first.

   \[
   \nabla f = \left\langle -\frac{1}{y} \sin\left(\frac{x}{y}\right), \frac{x}{y^2} \sin\left(\frac{x}{y}\right) \right\rangle
   \]

   **Step 2**
   Also recall that we need to make sure that the direction vector is a unit vector. It is (hopefully) pretty clear that this vector is not a unit vector so let’s convert it to a unit vector.
\[ \| \mathbf{v} \| = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5 \quad \mathbf{u} = \frac{\mathbf{v}}{\| \mathbf{v} \|} = \frac{1}{5} \langle 3, -4 \rangle = \langle \frac{3}{5}, -\frac{4}{5} \rangle \]

Step 3
The directional derivative is then,
\[
D_{\mathbf{u}} f = \langle -\frac{1}{y} \sin \left( \frac{x}{y} \right), -\frac{x}{y^2} \sin \left( \frac{x}{y} \right) \rangle \cdot \langle \frac{3}{5}, -\frac{4}{5} \rangle \\
= -\frac{3}{5y} \sin \left( \frac{x}{y} \right) - \frac{4x}{5y^2} \sin \left( \frac{x}{y} \right) = -\frac{1}{5} \left( \frac{3 + 4x}{y} \right) \sin \left( \frac{x}{y} \right)
\]

4. Determine \( D_{\mathbf{u}} f \) for \( f(x, y, z) = x^2 y^3 - 4xz \) in the direction of \( \mathbf{v} = \langle -1, 2, 0 \rangle \).

Step 1
Okay, we know we need the gradient so let’s get that first.
\[
\nabla f = \langle 2xy^3 - 4z, 3x^2 y^2, -4x \rangle
\]

Step 2
Also recall that we need to make sure that the direction vector is a unit vector. It is (hopefully) pretty clear that this vector is not a unit vector so let’s convert it to a unit vector.
\[
\| \mathbf{v} \| = \sqrt{(-1)^2 + (2)^2 + (0)^2} = \sqrt{5} \quad \mathbf{u} = \frac{\mathbf{v}}{\| \mathbf{v} \|} = \frac{1}{\sqrt{5}} \langle -1, 2, 0 \rangle = \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0 \right\rangle
\]

Step 3
The directional derivative is then,
\[
D_{\mathbf{u}} f = \langle 2xy^3 - 4z, 3x^2 y^2, -4x \rangle \cdot \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0 \right\rangle = \frac{1}{\sqrt{5}} \left( 4z - 2xy^3 + 6x^2 y^2 \right)
\]

5. Determine \( D_{\mathbf{u}} f \left( 3, -1, 0 \right) \) for \( f(x, y, z) = 4x - y^2 e^{3xz} \) direction of \( \mathbf{v} = \langle -1, 4, 2 \rangle \).

Step 1
Okay, we know we need the gradient so let’s get that first.
\[ \nabla f = \left\langle 4 - 3yz^2e^{3xz}, -2ye^{3xz}, -3x y^2e^{3xz} \right\rangle \]

Because we also know that we’ll eventually need this evaluated at the point we may as well get that as well.

\[ \nabla f (3, -1, 0) = \left\langle 4, 2, -9 \right\rangle \]

Step 2
Also recall that we need to make sure that the direction vector is a unit vector. It is (hopefully) pretty clear that this vector is not a unit vector so let’s convert it to a unit vector.

\[ \left\| \mathbf{v} \right\| = \sqrt{(-1)^2 + (4)^2 + (2)^2} = \sqrt{21} \quad \mathbf{u} = \frac{\mathbf{v}}{\left\| \mathbf{v} \right\|} = \frac{1}{\sqrt{21}} \langle -1, 4, 2 \rangle = \left\langle -\frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}, \frac{2}{\sqrt{21}} \right\rangle \]

Step 3
The directional derivative is then,

\[ D_{\mathbf{u}} f (3, -1, 0) = \langle 4, 2, -9 \rangle \left\langle -\frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}, \frac{2}{\sqrt{21}} \right\rangle = -\frac{14}{\sqrt{21}} \]

6. Find the maximum rate of change of \( f(x, y) = \sqrt{x^2 + y^4} \) at \((-2, 3)\) and the direction in which this maximum rate of change occurs.

Step 1
First we’ll need the gradient and its value at \((-2, 3)\).

\[ \nabla f = \left\langle \frac{x}{\sqrt{x^2 + y^4}}, \frac{2y^3}{\sqrt{x^2 + y^4}} \right\rangle \quad \nabla f (-2, 3) = \left\langle -\frac{2}{\sqrt{85}}, \frac{54}{\sqrt{85}} \right\rangle \]

Step 2
Now, by the theorem in class we know that the direction in which the maximum rate of change at the point in question is simply the gradient at \((-2, 3)\), which we found in the previous step. So, the direction in which the maximum rate of change of the function occurs is,

\[ \nabla f (-2, 3) = \left\langle -\frac{2}{\sqrt{85}}, \frac{54}{\sqrt{85}} \right\rangle \]

Step 3
The maximum rate of change is simply the magnitude of the gradient in the previous step. So, the maximum rate of change of the function is,
7. Find the maximum rate of change of \( f(x, y, z) = e^{2x} \cos(y - 2z) \) at \((4, -2, 0)\) and the direction in which this maximum rate of change occurs.

**Step 1**
First we’ll need the gradient and its value at \((4, -2, 0)\).

\[
\nabla f = \left\langle 2e^{2x} \cos(y - 2z), -e^{2x} \sin(y - 2z), 2e^{2x} \sin(y - 2z) \right\rangle
\]

\[
\nabla f(4, -2, 0) = \left\langle 2e^8 \cos(-2), -e^8 \sin(-2), 2e^8 \sin(-2) \right\rangle = \langle -2481.03, 2710.58, -5421.15 \rangle
\]

**Step 2**
Now, by the theorem in class we know that the direction in which the maximum rate of change at the point in question is simply the gradient at \((4, -2, 0)\), which we found in the previous step. So, the direction in which the maximum rate of change of the function occurs is,

\[
\nabla f(4, -2, 0) = \langle -2481.03, 2710.58, -5421.15 \rangle
\]

**Step 3**
The maximum rate of change is simply the magnitude of the gradient in the previous step. So, the maximum rate of change of the function is,

\[
\left\| \nabla f(4, -2, 0) \right\| = \sqrt{(-2481.03)^2 + (2710.58)^2 + (-5421.15)^2} = 6549.17
\]