Preface

Here are my online notes for my Calculus III course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Calculus III or needing a refresher in some of the topics from the class.

These notes do assume that the reader has a good working knowledge of Calculus I topics including limits, derivatives and integration. It also assumes that the reader has a good knowledge of several Calculus II topics including some integration techniques, parametric equations, vectors, and knowledge of three dimensional space.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Calculus III many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don’t have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren’t worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
Multiple Integrals

Introduction

In Calculus I we moved on to the subject of integrals once we had finished the discussion of derivatives. The same is true in this course. Now that we have finished our discussion of derivatives of functions of more than one variable we need to move on to integrals of functions of two or three variables.

Most of the derivatives topics extended somewhat naturally from their Calculus I counterparts and that will be the same here. However, because we are now involving functions of two or three variables there will be some differences as well. There will be new notation and some new issues that simply don’t arise when dealing with functions of a single variable.

Here is a list of topics covered in this chapter.

Double Integrals – We will define the double integral in this section.

Iterated Integrals – In this section we will start looking at how we actually compute double integrals.

Double Integrals over General Regions – Here we will look at some general double integrals.

Double Integrals in Polar Coordinates – In this section we will take a look at evaluating double integrals using polar coordinates.

Triple Integrals – Here we will define the triple integral as well as how we evaluate them.

Triple Integrals in Cylindrical Coordinates – We will evaluate triple integrals using cylindrical coordinates in this section.

Triple Integrals in Spherical Coordinates – In this section we will evaluate triple integrals using spherical coordinates.

Change of Variables – In this section we will look at change of variables for double and triple integrals.

Surface Area – Here we look at the one real application of double integrals that we’re going to look at in this material.

Area and Volume Revisited – We summarize the area and volume formulas from this chapter.
**Double Integrals**

Before starting on double integrals let’s do a quick review of the definition of a definite integral for functions of single variables. First, when working with the integral,

$$\int_a^b f(x) \, dx$$

we think of $x$’s as coming from the interval $a \leq x \leq b$. For these integrals we can say that we are integrating over the interval $a \leq x \leq b$. Note that this does assume that $a < b$, however, if we have $b < a$ then we can just use the interval $b \leq x \leq a$.

Now, when we derived the definition of the definite integral we first thought of this as an area problem. We first asked what the area under the curve was and to do this we broke up the interval $a \leq x \leq b$ into $n$ subintervals of width $\Delta x$ and choose a point, $x_i^*$, from each interval as shown below,

Each of the rectangles has height of $f(x_i^*)$ and we could then use the area of each of these rectangles to approximate the area as follows.

$$A \approx f(x_1^*) \Delta x + f(x_2^*) \Delta x + \cdots + f(x_n^*) \Delta x$$

To get the exact area we then took the limit as $n$ goes to infinity and this was also the definition of the definite integral.

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x$$

In this section we want to integrate a function of two variables, $f(x, y)$. With functions of one variable we integrated over an interval (i.e. a one-dimensional space) and so it makes some sense then that when integrating a function of two variables we will integrate over a region of $\mathbb{R}^2$ (two-dimensional space).
We will start out by assuming that the region in $\mathbb{R}^2$ is a rectangle which we will denote as follows,

$$R = [a, b] \times [c, d]$$

This means that the ranges for $x$ and $y$ are $a \leq x \leq b$ and $c \leq y \leq d$.

Also, we will initially assume that $f(x, y) \geq 0$ although this doesn’t really have to be the case.

Let’s start out with the graph of the surface $S$ given by graphing $f(x, y)$ over the rectangle $R$.

Now, just like with functions of one variable let’s not worry about integrals quite yet. Let’s first ask what the volume of the region under $S$ (and above the $xy$-plane of course) is.

We will first approximate the volume much as we approximated the area above. We will first divide up $a \leq x \leq b$ into $n$ subintervals and divide up $c \leq y \leq d$ into $m$ subintervals. This will divide up $R$ into a series of smaller rectangles and from each of these we will choose a point $(x^*, y^*)$. Here is a sketch of this set up.
Now, over each of these smaller rectangles we will construct a box whose height is given by $f(x^*_i, y^*_j)$. Here is a sketch of that.

Each of the rectangles has a base area of $\Delta A$ and a height of $f(x^*_i, y^*_j)$ so the volume of each of these boxes is $f(x^*_i, y^*_j)\Delta A$. The volume under the surface $S$ is then approximately,

$$V \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f(x^*_i, y^*_j)\Delta A$$

We will have a double sum since we will need to add up volumes in both the $x$ and $y$ directions.

To get a better estimation of the volume we will take $n$ and $m$ larger and larger and to get the exact volume we will need to take the limit as both $n$ and $m$ go to infinity. In other words,
Now, this should look familiar. This looks a lot like the definition of the integral of a function of single variable. In fact this is also the definition of a double integral, or more exactly an integral of a function of two variables over a rectangle.

Here is the official definition of a double integral of a function of two variables over a rectangular region $R$ as well as the notation that we’ll use for it.

$$\iint_{R} f(x, y) \, dA = \lim_{n,m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{i}^{*}, y_{j}^{*}) \Delta A$$

Note the similarities and differences in the notation to single integrals. We have two integrals to denote the fact that we are dealing with a two dimensional region and we have a differential here as well. Note that the differential is $dA$ instead of the $dx$ and $dy$ that we’re used to seeing. Note as well that we don’t have limits on the integrals in this notation. Instead we have the $R$ written below the two integrals to denote the region that we are integrating over.

Note that one interpretation of the double integral of $f(x, y)$ over the rectangle $R$ is the volume under the function $f(x, y)$ (and above the $xy$-plane). Or,

$$\text{Volume} = \iint_{R} f(x, y) \, dA$$

We can use this double sum in the definition to estimate the value of a double integral if we need to. We can do this by choosing $\left(x_{i}^{*}, y_{j}^{*}\right)$ to be the midpoint of each rectangle. When we do this we usually denote the point as $\left(\bar{x}_{i}, \bar{y}_{j}\right)$. This leads to the Midpoint Rule,

$$\iint_{R} f(x, y) \, dA \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f(\bar{x}_{i}, \bar{y}_{j}) \Delta A$$

In the next section we start looking at how to actually compute double integrals.