Preface

Here are my online notes for my Calculus III course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Calculus III or needing a refresher in some of the topics from the class.

These notes do assume that the reader has a good working knowledge of Calculus I topics including limits, derivatives and integration. It also assumes that the reader has a good knowledge of several Calculus II topics including some integration techniques, parametric equations, vectors, and knowledge of three dimensional space.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Calculus III many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don’t have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren’t worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
In this section we will look at the lone application (aside from the area and volume interpretations) of multiple integrals in this material. This is not the first time that we’ve looked at surface area. We first saw surface area in Calculus II, however, in that setting we were looking at the surface area of a solid of revolution. In other words we were looking at the surface area of a solid obtained by rotating a function about the $x$ or $y$ axis. In this section we want to look at a much more general setting although you will note that the formula here is very similar to the formula we saw back in Calculus II.

Here we want to find the surface area of the surface given by $z = f(x, y)$ where $(x, y)$ is a point from the region $D$ in the $xy$-plane. In this case the surface area is given by,

$$ S = \iint_D \sqrt{[f_x]^2 + [f_y]^2 + 1} \, dA $$

Let’s take a look at a couple of examples.

**Example 1** Find the surface area of the part of the plane $3x + 2y + z = 6$ that lies in the first octant.

**Solution**

Remember that the first octant is the portion of the $xyz$-axis system in which all three variables are positive. Let’s first get a sketch of the part of the plane that we are interested in.

We’ll also need a sketch of the region $D$. 
Remember that to get the region $D$ we can pretend that we are standing directly over the plane and what we see is the region $D$. We can get the equation for the hypotenuse of the triangle by realizing that this is nothing more than the line where the plane intersects the $xy$-plane and we also know that $z = 0$ on the $xy$-plane. Plugging $z = 0$ into the equation of the plane will give us the equation for the hypotenuse.

Notice that in order to use the surface area formula we need to have the function in the form $z = f(x, y)$ and so solving for $z$ and taking the partial derivatives gives,

$$z = 6 - 3x - 2y$$

$$f_x = -3$$

$$f_y = -2$$

The limits defining $D$ are,

$$0 \leq x \leq 2$$

$$0 \leq y \leq -\frac{3}{2}x + 3$$

The surface area is then,

$$S = \iint_D \sqrt{f_x^2 + f_y^2} + 1 \, dA$$

$$= \int_0^2 \int_0^{-\frac{3}{2}x + 3} \sqrt{14} \, dy \, dx$$

$$= \sqrt{14} \int_0^2 -\frac{3}{2}x + 3 \, dx$$

$$= \sqrt{14} \left( -\frac{3}{4}x^2 + 3x \right)_0^2$$

$$= 3\sqrt{14}$$

**Example 2**: Determine the surface area of the part of $z = xy$ that lies in the cylinder given by $x^2 + y^2 = 1$.

**Solution**

In this case we are looking for the surface area of the part of $z = xy$ where $(x, y)$ comes from the disk of radius 1 centered at the origin since that is the region that will lie inside the given
cylinder.

Here are the partial derivatives,

\[ f_x = y \quad f_y = x \]

The integral for the surface area is,

\[ S = \iint_D \sqrt{x^2 + y^2 + 1} \, dA \]

Given that \( D \) is a disk it makes sense to do this integral in polar coordinates.

\[ S = \iint_D \sqrt{x^2 + y^2 + 1} \, dA \]

\[ = \int_0^{2\pi} \int_0^1 r \sqrt{1 + r^2} \, dr \, d\theta \]

\[ = \int_0^{2\pi} \left[ \frac{1}{2} \left( \frac{2}{3} \right) \left( 1 + r^2 \right)^{\frac{3}{2}} \right]_0^1 \, d\theta \]

\[ = \int_0^{2\pi} \frac{1}{3} \left( 2^3 - 1 \right) \, d\theta \]

\[ = \frac{2\pi}{3} \left( 2^3 - 1 \right) \]