Preface

Here are the solutions to the practice problems for my Calculus II notes. Some solutions will have more or less detail than other solutions. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you’ve reached the level of working the harder problems then you will probably already understand the basics fairly well and won’t need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven’t been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.
Fundamental Theorem for Line Integrals

1. Evaluate \( \int_C \nabla f \cdot d\vec{r} \) where \( f(x,y) = x^3 (3 - y^2) + 4y \) and \( C \) is given by \( \vec{r}(t) = \langle 3 - t^2, 5 - t \rangle \) with \(-2 \leq t \leq 3\).

Step 1
There really isn’t all that much to do with this problem. We are integrating over a gradient vector field and so the integral is set up to use the Fundamental Theorem for Line Integrals.

To do that we’ll need the following two “points”.

\[
\vec{r}(-2) = \langle -1, 7 \rangle \quad \text{and} \quad \vec{r}(3) = \langle -6, 2 \rangle
\]

Remember that we are thinking of these as the position vector representations of the points \((-1,7)\) and \((-6,2)\) respectively.

Step 2
Now simply apply the Fundamental Theorem to evaluate the integral.

\[
\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(3)) - f(\vec{r}(-2)) = f(-6,2) - f(-1,7) = 224 - 74 = 150
\]

2. Evaluate \( \int_C \nabla f \cdot d\vec{r} \) where \( f(x,y) = ye^{x^2 - 1} + 4x\sqrt{y} \) and \( C \) is given by \( \vec{r}(t) = \langle 1 - t, 2t^2 - 2t \rangle \) with \(0 \leq t \leq 2\).

Step 1
There really isn’t all that much to do with this problem. We are integrating over a gradient vector field and so the integral is set up to use the Fundamental Theorem for Line Integrals.

To do that we’ll need the following two “points”.

\[
\vec{r}(0) = \langle 1, 0 \rangle \quad \text{and} \quad \vec{r}(2) = \langle -1, 4 \rangle
\]

Remember that we are thinking of these as the position vector representations of the points \((1,0)\) and \((-1,4)\) respectively.

Step 2
Now simply apply the Fundamental Theorem to evaluate the integral.
\[ \int_{C} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(2)) - f(\mathbf{r}(0)) = f(-1,4) - f(1,0) = -4 - 0 = -4 \]

3. Given that \( \int_{C} \mathbf{F} \cdot d\mathbf{r} \) is independent of path compute \( \int_{C} \mathbf{F} \cdot d\mathbf{r} \) where \( C \) is the ellipse given by
\[
\frac{(x-5)^2}{4} + \frac{y^2}{9} = 1 \text{ with the counter clockwise rotation.}
\]

Solution
At first glance this problem seems to be impossible since the vector field isn’t even given for the problem. However, it’s actually quite simple and the vector field is not needed to do the problem.

There are two important things to note in the problem statement.

First, and somewhat more importantly, we are told in the problem statement that the integral is independent of path.

Second we are told that the curve, \( C \), is the full ellipse. It isn’t the fact that \( C \) is an ellipse that is important. What is important is the fact that \( C \) is a closed curve.

Now all we need to do is use Fact 4 from the notes. This tells us that the value of a line integral of this type around a closed path will be zero if the integral is independent of path. Therefore,
\[ \int_{C} \mathbf{F} \cdot d\mathbf{r} = 0 \]

4. Evaluate \( \int_{C} \nabla f \cdot d\mathbf{r} \) where \( f(x,y) = e^{xy} - x^2 + y^3 \) and \( C \) is the curve shown below.
Solution
This problem is much simpler than it appears at first. We do not need to compute 3 different line integrals (one for each curve in the sketch).

All we need to do is notice that we are doing a line integral for a gradient vector function and so we can use the Fundamental Theorem for Line Integrals to do this problem.

Using the Fundamental Theorem to evaluate the integral gives the following,

\[
\int_C \nabla f \cdot d\mathbf{r} = f(\text{end point}) - f(\text{start point})
\]

\[
= f(0,-2) - f(-2,0)
\]

\[
= -7 - (-3) = 4
\]

Remember that all the Fundamental Theorem requires is the starting and ending point of the curve and the function used to generate the gradient vector field.