Preface

Here are the solutions to the practice problems for my Calculus II notes. Some solutions will have more or less detail than other solutions. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you’ve reached the level of working the harder problems then you will probably already understand the basics fairly well and won’t need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven’t been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.
Conservative Vector Fields

1. Determine if the following vector field is conservative.

\[ \vec{F} = \left( x^3 - 4xy^2 + 2 \right) \hat{i} + \left( 6x - 7y + x^3y^3 \right) \hat{j} \]

Solution
There really isn’t all that much to do with this problem. All we need to do is identify \( P \) and \( Q \) then run through the test.

So,

\[
\begin{align*}
P &= x^3 - 4xy^2 + 2 \\
Q &= 6x - 7y + x^3y^3
\end{align*}
\]

\[
\begin{align*}
P_y &= -8xy \\
Q_x &= 6 + 3x^2y^3
\end{align*}
\]

Okay, we can clearly see that \( P_y \neq Q_x \) and so the vector field is **not conservative**.

2. Determine if the following vector field is conservative.

\[ \vec{F} = \left( 2x \sin(2y) - 3y^2 \right) \hat{i} + \left( 2 - 6xy + 2x^2 \cos(2y) \right) \hat{j} \]

Solution
There really isn’t all that much to do with this problem. All we need to do is identify \( P \) and \( Q \) then run through the test.

So,

\[
\begin{align*}
P &= 2x \sin(2y) - 3y^2 \\
Q &= 2 - 6xy + 2x^2 \cos(2y)
\end{align*}
\]

\[
\begin{align*}
P_y &= 4x \cos(2y) - 6y \\
Q_x &= -6y + 4x \cos(2y)
\end{align*}
\]

Okay, we can clearly see that \( P_y = Q_x \) and so the vector field is **conservative**.

3. Determine if the following vector field is conservative.

\[ \vec{F} = \left( 6 - 2xy + y^3 \right) \hat{i} + \left( x^2 - 8y + 3xy^2 \right) \hat{j} \]

Solution
There really isn’t all that much to do with this problem. All we need to do is identify \( P \) and \( Q \) then run through the test.

So,
Okay, we can clearly see that \( P_y \neq Q_x \) and so the vector field is **not conservative**.

Be careful with these problems. It is easy to get into a hurry and miss a very subtle difference between the two derivatives. In this case, the only difference between the two derivatives is the sign on the first term. That’s it. That is also enough for this vector field to not be conservative.

4. Find the potential function for the following vector field.

\[
\vec{F} = \left( 6x^2 - 2xy^2 + \frac{y}{2\sqrt{x}} \right) \hat{i} - \left( 2x^2y - 4 - \sqrt{x} \right) \hat{j}
\]

**Step 1**

Now, by assumption from how the problem was asked, we could assume that the vector field is conservative but let’s check it anyway just to make sure.

So,

\[
P = 6x^2 - 2xy^2 + \frac{y}{2\sqrt{x}} \quad P_y = -4xy + \frac{1}{2\sqrt{x}}
\]

\[
Q = -\left( 2x^2y - 4 - \sqrt{x} \right) \quad Q_x = -4xy + 4 + \frac{1}{2\sqrt{x}}
\]

Okay, we can see that \( P_y = Q_x \) and so the vector field is conservative as the problem statement suggested it would be.

Be careful with these problems and watch the signs on the vector components. One of the biggest mistakes that students make with these problems is to miss the minus sign that is in front of the second component of the vector field. There won’t always be a minus sign of course, but on occasion there will be one and if we miss it the rest of the problem will be very difficult to do. In fact, if we miss it we won’t be able to find a potential function for the vector field!

**Step 2**

Okay, to find the potential function for this vector field we know that we need to first either integrate \( P \) with respect to \( x \) or integrate \( Q \) with respect to \( y \). It doesn’t matter which one we use chose to use and, in this case, it looks like neither will be any harder than the other.

So, let’s go with the following integration for this problem.
\[ f(x, y) = \int Q \, dy \]
\[ = \int -2x^2y + 4 + \sqrt{x} \, dy \]
\[ = -x^2y^2 + 4y + y\sqrt{x} + g(x) \]

Don’t forget that, in this case, because we were integrating with respect to \( y \) the “constant of integration” will be a function of \( x \)!

Step 3
Next, differentiate the function from the previous step with respect to \( x \) and set equal to \( P \) since we know the derivative and \( P \) should be the same function.

\[ f_x = -2xy^2 + \frac{y}{2\sqrt{x}} + g'(x) = 6x^2 - 2xy^2 + \frac{y}{2\sqrt{x}} = P \quad \Rightarrow \quad g'(x) = 6x^2 \]

Now, recall that because we integrated with respect to \( y \) in Step 2 \( g(x) \), and hence \( g'(x) \), should only be a function of \( x \)’s (as it is in this case). If there had been any \( y \)’s in \( g'(x) \) we’d know there was something wrong at this point. Either we’d made a mistake somewhere or the vector field was not conservative. Of course we verified that it was conservative in Step 1 and so this would in fact mean we’d made a mistake somewhere!

Step 4
We can now integrate both sides of the formula for \( g'(x) \) above to get,

\[ g(x) = 2x^3 + c \]

Don’t forget the “ + \( c \)” on this!

Step 5
Finally, putting everything together we get the following potential function for the vector field.

\[ f(x, y) = -x^2y^2 + 4y + y\sqrt{x} + 2x^3 + c \]

5. Find the potential function for the following vector field.

\[ \vec{F} = y^2 \left( 1 + \cos(x + y) \right) \hat{i} + \left( 2xy - 2y + y^2 \cos(x + y) + 2y \sin(x + y) \right) \hat{j} \]

Step 1
Now, by assumption from how the problem was asked, we could assume that the vector field is conservative but let’s check it anyway just to make sure.

So,
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\[ P = y^2 \left( 1 + \cos(x + y) \right) = y^2 + y^2 \cos(x + y) \quad P_y = 2y - y^2 \sin(x + y) + 2y \cos(x + y) \]

\[ Q = 2xy - 2y + y^2 \cos(x + y) + 2y \sin(x + y) \quad Q_x = 2y - y^2 \sin(x + y) + 2y \cos(x + y) \]

Okay, we can see that \( P_y = Q_x \) and so the vector field is conservative as the problem statement suggested it would be.

Step 2
Okay, to find the potential function for this vector field we know that we need to first either integrate \( P \) with respect to \( x \) or integrate \( Q \) with respect to \( y \). It doesn’t matter which one we use chose to use in general, but in in this case integrating \( Q \) with respect to \( y \) just looks painful (two integration by parts terms!).

So, let’s go with the following integration for this problem.

\[
\begin{align*}
 f(x, y) &= \int P \, dx \\
 &= \int (y^2 + y^2 \cos(x + y)) \, dx \\
 &= xy^2 + y^2 \sin(x + y) + h(y)
\end{align*}
\]

Don’t forget that, in this case, because we were integrating with respect to \( x \) the “constant of integration” will be a function of \( y \)!

Note, that as this problem has shown, sometimes one integration order will be significantly easier than the other so be on the lookout for which term might be easier to integrate.

Step 3
Next, differentiate the function from the previous step with respect to \( y \) and set equal to \( Q \) since we know the derivative and \( Q \) should be the same function.

\[
\begin{align*}
 f_y &= 2xy + 2y \sin(x + y) + y^2 \cos(x + y) + h'(y) \\
 &= 2xy - 2y + y^2 \cos(x + y) + 2y \sin(x + y) = Q \\
 \Rightarrow h'(y) &= -2y
\end{align*}
\]

Now, recall that because we integrated with respect to \( x \) in Step 2 \( h(y) \), and hence \( h'(y) \), should only be a function of \( y \’s \) (as it is in this case). If there had been any \( x \’s \) in \( h'(y) \) we’d know there was something wrong at this point. Either we’d made a mistake somewhere or the vector field was not conservative. Of course we verified that it was conservative in Step 1 and so this would in fact mean we’d made a mistake somewhere!

Step 4
We can now integrate both sides of the formula for \( h'(y) \) above to get,

\[ h(y) = -y^2 + c \]
Don’t forget the “ + c” on this!

Step 5
Finally, putting everything together we get the following potential function for the vector field.

\[ f(x, y) = x y^2 + y^2 \sin(x + y) - y^2 + c \]

6. Find the potential function for the following vector field.

\[ \vec{F} = (2z^4 - 2y - y^3)\hat{i} + (z - 2x - 3xy^2)\hat{j} + (6 + y + 8xz^3)\hat{k} \]

Step 1
Now, by assumption from how the problem was asked, we can assume that the vector field is conservative and because we don’t know how to verify this for a 3D vector field we will just need to trust that it is.

Let’s start off the problem by labeling each of the components to make the problem easier to deal with as follows.

\[
\begin{align*}
P &= 2z^4 - 2y - y^3 \\
Q &= z - 2x - 3xy^2 \\
R &= 6 + y + 8xz^3
\end{align*}
\]

Step 2
To find the potential function for this vector field we know that we need to first either integrate \(P\) with respect to \(x\), integrate \(Q\) with respect to \(y\) or \(R\) with respect to \(z\). It doesn’t matter which one we use chose to use and, in this case, it looks like none of them will be any harder than the others.

So, let’s go with the following integration for this problem.

\[
f(x, y, z) = \int Q \, dy = \int (z - 2x - 3xy^2) \, dy = zy - 2xy - xy^3 + h(x, z)
\]

Don’t forget that, in this case, because we were integrating with respect to \(y\) the “constant of integration” will be a function of \(x\) and/or \(z\)!

Step 3
Next, we can differentiate the function from the previous step with respect to \(x\) and set equal to \(P\) or differentiate the function with respect to \(z\) and set equal to \(R\).

Again neither looks any more difficult than the other so let’s differentiate with respect to \(z\).
\[ f_z = y + h_z (x, z) = 6 + y + 8xz^3 = R \quad \Rightarrow \quad h_z (x, z) = 6 + 8xz^3 \]

Now, recall that because we integrated with respect to \( y \) in Step 2 \( h(x, z) \), and hence \( h_z (x, z) \), should only be a function of \( x \)'s and \( z \)'s (as it is in this case). If there had been any \( y \)'s in \( h_z (x, z) \) we'd know there was something wrong at this point. Either we'd made a mistake somewhere or the vector field was not conservative.

Also note that there is no reason to expect \( h_z (x, z) \) to have both \( x \)'s and \( z \)'s in it. It is completely possible for one (or both) of the variables to not be present!

Step 4
We can now integrate both sides of the formula for \( h_z (x, z) \) with respect to \( z \) to get,

\[ h(x, z) = 6z + 2xz^4 + g(x) \]

Now, because \( h(x, z) \) was a function of both \( x \) and \( z \) and we integrated with respect to \( z \) here the “constant of integration” in this case would need to be a function of \( x \), \( g(x) \) in this case.

The potential function is now,

\[ f(x, y, z) = zy - 2xy - xy^3 + 6z + 2xz^4 + g(x) \]

Step 5
Next, we’ll need to differentiate the potential function from Step 4 with respect to \( x \) and set equal to \( P \). Doing this gives,

\[ f_x = -2y - y^3 + 2z^4 + g'(x) = 2z^4 - 2y - y^3 = P \quad \Rightarrow \quad g'(x) = 0 \]

Remember, that as in Step 3, we have to recall what variable we are differentiating with respect to here. In this case we are differentiating with respect to \( x \) and so \( g(x) \) should only be a function of \( x \). Had \( g'(x) \) contained either \( y \)'s or \( z \)'s we’d know that either we’d made a mistake or the vector field was not conservative.

Also, as shown in this problem, it is completely possible for there to be no \( x \)'s at all in \( g'(x) \).

Step 6
Integrating both sides of the formula for \( g'(x) \) from Step 5 and we can see that we must have \( g(x) = c \).

Step 7
Finally, putting everything together we get the following potential function for the vector field.
7. Find the potential function for the following vector field.

\[ \vec{F} = \frac{2xy}{z^3} \hat{i} + \left(2y - z^2 + \frac{x^2}{z^3}\right) \hat{j} - \left(4z^3 + 2yz + \frac{3x^2y}{z^4}\right) \hat{k} \]

Step 1
Now, by assumption from how the problem was asked, we can assume that the vector field is conservative and because we don’t know how to verify this for a 3D vector field we will just need to trust that it is.

Let’s start off the problem by labeling each of the components to make the problem easier to deal with as follows.

\[
\begin{align*}
P &= \frac{2xy}{z^3} \\
Q &= 2y - z^2 + \frac{x^2}{z^3} \\
R &= -\left(4z^3 + 2yz + \frac{3x^2y}{z^4}\right) = -4z^3 - 2yz - \frac{3x^2y}{z^4}
\end{align*}
\]

Be careful with these problems and watch the signs on the vector components. One of the biggest mistakes that students make with these problems is to miss the minus sign that is in front of the third component of the vector field. There won’t always be a minus sign of course, but on occasion there will be one and if we miss it the rest of the problem will be very difficult to do. In fact, if we miss it we won’t be able to find a potential function for the vector field!

Step 2
To find the potential function for this vector field we know that we need to first either integrate \( P \) with respect to \( x \), integrate \( Q \) with respect to \( y \) or \( R \) with respect to \( z \). It doesn’t matter which one we use chose to use and, in this case, it looks like none of them will be any harder than the other.

In this case the \( R \) has quite a few terms in it so let’s integrate that one first simply because it might mean less work when dealing with \( P \) and \( Q \) in later steps.

\[
f(x, y, z) = \int R \, dz = \int \left(-4z^3 - 2yz - \frac{3x^2y}{z^4}\right) \, dz = -z^4 - yz^2 + x^2yz^{-3} + h(x, y)
\]

Don’t forget that, in this case, because we were integrating with respect to \( z \) the “constant of integration” will be a function of \( x \) and/or \( y \)!
Step 3
Next, we can differentiate the function from the previous step with respect to $x$ and set equal to $P$ or differentiate the function with respect to $y$ and set equal to $Q$.

Let’s differentiate with respect to $y$ in this case.

\[ f_y = -z^2 + x^2z^{-3} + h_y(x, y) = 2y - z^2 + x^2z^{-3} = Q \quad \Rightarrow \quad h_y(x, y) = 2y \]

Now, recall that because we integrated with respect to $z$ in Step 2 $h(x, y)$, and hence $h_y(x, y)$, should only be a function of $x$’s and $y$’s (as it is in this case). If there had been any $z$’s in $h_y(x, y)$ we’d know there was something wrong at this point. Either we’d made a mistake somewhere or the vector field was not conservative.

Also note that there is no reason to expect $h_y(x, y)$ to have both $x$’s and $y$’s in it. It is completely possible for one (or both) of the variables to not be present!

Step 4
We can now integrate both sides of the formula for $h_y(x, y)$ with respect to $y$ to get,

\[ h(x, y) = y^2 + g(x) \]

Now, because $h(x, y)$ was a function of both $x$ and $y$ and we integrated with respect to $y$ here the “constant of integration” in this case would need to be a function of $x$, $g(x)$ in this case.

The potential function is now,

\[ f(x, y, z) = -z^4 - yz^2 + x^2yz^{-3} + y^2 + g(x) \]

Step 5
Next, we’ll need to differentiate the potential function from Step 4 with respect to $x$ and set equal to $P$. Doing this gives,

\[ f_x = 2xyz^{-3} + g'(x) = 2xyz = P \quad \Rightarrow \quad g'(x) = 0 \]

Remember, that as in Step 3, we have to recall what variable we are differentiating with respect to here. In this case we are differentiating with respect to $x$ and $g(x)$ should only be a function of $x$. Had $g'(x)$ contained either $y$’s or $z$’s we’d know that either we’d made a mistake or the vector field was not conservative.

Also, as shown in this problem it is completely possible for there to be no $x$’s at all in $g'(x)$.

Step 6
Integrating both sides of the formula for \( g'(x) \) from Step 5 and we can see that we must have \( g(x) = c \).

**Step 7**
Finally, putting everything together we get the following potential function for the vector field.

\[
f(x, y, z) = -z^4 - yz^2 + x^2 y z^3 + y^2 + c
\]

8. Evaluate \( \int_C \vec{F} \cdot d\vec{r} \) where \( C \) is the portion of the circle centered at the origin with radius 2 in the 1st quadrant with counter clockwise rotation and

\[
\vec{F}(x, y) = \left(2xy - \frac{1}{2} \sin \left(\frac{1}{2} x\right) \sin \left(\frac{1}{2} y\right)\right)i + \left(x^2 + \frac{1}{2} \cos \left(\frac{1}{2} x\right) \cos \left(\frac{1}{2} y\right)\right)j.
\]

**Step 1**
There are two ways to work this problem, a hard way and an easy way. The hard way is to just see a line integral with a curve and a vector field given and just launch into computing the line integral directly (probably very difficult in this case). The easy way is to check and see if the vector field is conservative, and if it is find the potential function and then simply use the Fundamental Theorem for Line Integrals that we saw in the previous section.

So, let’s go the easy way and check to see if the vector field is conservative.

\[
P = 2xy - \frac{1}{2} \sin \left(\frac{1}{2} x\right) \sin \left(\frac{1}{2} y\right) \quad P_y = 2x - \frac{1}{4} \sin \left(\frac{1}{2} x\right) \cos \left(\frac{1}{2} y\right)
\]

\[
Q = x^2 + \frac{1}{2} \cos \left(\frac{1}{2} x\right) \cos \left(\frac{1}{2} y\right) \quad Q_x = 2x - \frac{1}{4} \sin \left(\frac{1}{2} x\right) \cos \left(\frac{1}{2} y\right)
\]

So, we can see that \( P_y = Q_x \) and so the vector field is conservative.

**Step 2**
Now we just need to find the potential function for the vector field. We’ll go through those details a little quicker this time and with less explanation than we did in the some of the previous problems.

First, let’s integrate \( P \) with respect to \( x \).

\[
f(x, y) = \int P \, dx
\]

\[
= \int 2xy - \frac{1}{2} \sin \left(\frac{1}{2} x\right) \sin \left(\frac{1}{2} y\right) \, dx
\]

\[
= x^2 y - 4x + \cos \left(\frac{1}{2} x\right) \sin \left(\frac{1}{2} y\right) + h(y)
\]

Now, differentiate with respect to \( y \) and set equal to \( Q \).

\[
f_y = x^2 + \frac{1}{2} \cos \left(\frac{1}{2} x\right) \cos \left(\frac{1}{2} y\right) + h'(y) = x^2 + \frac{1}{2} \cos \left(\frac{1}{2} x\right) \cos \left(\frac{1}{2} y\right) = Q \quad \Rightarrow \quad h'(y) = 0
\]
Solving for \( h(y) \) gives \( h(y) = c \) and so the potential function for this vector field is,

\[
f(x, y) = x^2 y - 4x + \cos\left(\frac{1}{2} x\right) \sin\left(\frac{1}{2} y\right) + c
\]

Step 3
Now that we have the potential function we can simply use the Fundamental Theorem for Line Integrals which says,

\[
\int_C \vec{F} \cdot d\vec{r} = f(\text{end point}) - f(\text{start point})
\]

From the problem statement we know that \( C \) is the portion of the circle of radius 2 in the 1st quadrant with counter clockwise rotation. Therefore the starting point of \( C \) is \( (2, 0) \) and the ending point of \( C \) is \( (0, 2) \).

The integral is then,

\[
\int_C \vec{F} \cdot d\vec{r} = f(0, 2) - f(2, 0) = (\sin(1) + c) - (-8 + c) = \sin(1) + 8 = 8.8415
\]

9. Evaluate \( \int_C \vec{F} \cdot d\vec{r} \) where \( \vec{F}(x, y) = \left(2ye^{xy} + 2xe^{x^2-y^2}\right)\hat{i} + \left(2xe^{xy} - 2ye^{x^2-y^2}\right)\hat{j} \) and \( C \) is the curve shown below.

Step 1
There are two ways to work this problem, a hard way and an easy way. The hard way is to just see a line integral with a curve and a vector field given and just launch into computing the line integral directly (probably quite unpleasant in this case). The easy way is to check and see if the vector field is conservative, and if it is find the potential function and then simply use the Fundamental Theorem for Line Integrals that we saw in the previous section.

So, let’s go the easy way and check to see if the vector field is conservative.
So, we can see that \( P_y = Q_x \) and so the vector field is conservative.

Step 2
Now we just need to find the potential function for the vector field. We’ll go through those details a little quicker this time and with less explanation than we did in the some of the previous problems.

First, let’s integrate \( Q \) with respect to \( y \).

\[
f(x, y) = \int Q \, dy = \int 2xe^{xy} - 2ye^{x^2-y^2} \, dy = 2e^{xy} + e^{x^2-y^2} + g(x)
\]

Now, differentiate with respect to \( x \) and set equal to \( P \).

\[
f_x = 2ye^{xy} + 2xe^{x^2-y^2} + g'(x) = 2ye^{xy} + 2xe^{x^2-y^2} = P \quad \Rightarrow \quad g'(x) = 0
\]

Solving for \( g(x) \) gives \( g(x) = c \) and so the potential function for this vector field is,

\[
f(x, y) = 2e^{xy} + e^{x^2-y^2} + c
\]

Step 3
Now that we have the potential function we can simply use the Fundamental Theorem for Line Integrals which says,

\[
\int_{C} \vec{F} \cdot d\vec{r} = f(\text{end point}) - f(\text{start point})
\]

From the graph in the problem statement we can see that the starting point of \( C \) is \((0,1)\) and the ending point of \( C \) is \((5,0)\).

The integral is then,

\[
\int_{C} \vec{F} \cdot d\vec{r} = f(5,0) - f(0,1) = \left(2 + e^{25} + c\right) - \left(2 + e^{-1} + c\right) = e^{25} - e^{-1}
\]