Preface

Here are my online notes for my Calculus III course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Calculus III or needing a refresher in some of the topics from the class.

These notes do assume that the reader has a good working knowledge of Calculus I topics including limits, derivatives and integration. It also assumes that the reader has a good knowledge of several Calculus II topics including some integration techniques, parametric equations, vectors, and knowledge of three dimensional space.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Calculus III many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don’t have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren’t worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
Curl and Divergence

In this section we are going to introduce a couple of new concepts, the curl and the divergence of a vector.

Let’s start with the curl. Given the vector field \( \vec{F} = P \vec{i} + Q \vec{j} + R \vec{k} \) the curl is defined to be,

\[
\text{curl} \, \vec{F} = (R_y - Q_z) \vec{i} + (P_z - R_x) \vec{j} + (Q_x - P_y) \vec{k}
\]

There is another (potentially) easier definition of the curl of a vector field. To use it we will first need to define the gradient operator. This is defined to be,

\[
\nabla = \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k}
\]

We use this as if it’s a function in the following manner.

\[
\nabla f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}
\]

So, whatever function is listed after the \( \nabla \) is substituted into the partial derivatives. Note as well that when we look at it in this light we simply get the gradient vector.

Using the \( \nabla \) we can define the curl as the following cross product,

\[
\text{curl} \, \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}
\]

We have a couple of nice facts that use the curl of a vector field.

Facts

1. If \( f(x, y, z) \) has continuous second order partial derivatives then \( \text{curl} \, (\nabla f) = \vec{0} \). This is easy enough to check by plugging into the definition of the derivative so we’ll leave it to you to check.

2. If \( \vec{F} \) is a conservative vector field then \( \text{curl} \, \vec{F} = \vec{0} \). This is a direct result of what it means to be a conservative vector field and the previous fact.

3. If \( \vec{F} \) is defined on all of \( \mathbb{R}^3 \) whose components have continuous first order partial derivative and \( \text{curl} \, \vec{F} = \vec{0} \) then \( \vec{F} \) is a conservative vector field. This is not so easy to verify and so we won’t try.
Example 1  Determine if \( \vec{F} = x^2 y \hat{i} + xyz \hat{j} - x^2 y^2 \hat{k} \) is a conservative vector field.

Solution
So all that we need to do is compute the curl and see if we get the zero vector or not.

\[
\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & xyz & -x^2 y^2 \end{vmatrix} \\
= -2x^2 y \hat{i} + yz \hat{k} - \left( -2xy^2 \right) \hat{j} - xy \hat{i} - x^2 \hat{k} \\
= -\left( 2x^2 y + xy \right) \hat{i} + 2xy^2 \hat{j} + \left( yz - x^2 \right) \hat{k} \\
\neq \hat{0}
\]

So, the curl isn’t the zero vector and so this vector field is not conservative.

Next we should talk about a physical interpretation of the curl. Suppose that \( \vec{F} \) is the velocity field of a flowing fluid. Then \( \text{curl } \vec{F} \) represents the tendency of particles at the point \((x, y, z)\) to rotate about the axis that points in the direction of \( \text{curl } \vec{F} \). If \( \text{curl } \vec{F} = \hat{0} \) then the fluid is called irrotational.

Let’s now talk about the second new concept in this section. Given the vector field \( \vec{F} = P \hat{i} + Q \hat{j} + R \hat{k} \) the divergence is defined to be,

\[
\text{div } \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}
\]

There is also a definition of the divergence in terms of the \( \nabla \) operator. The divergence can be defined in terms of the following dot product.

\[
\text{div } \vec{F} = \nabla \cdot \vec{F}
\]

Example 2  Compute \( \text{div } \vec{F} \) for \( \vec{F} = x^2 y \hat{i} + xyz \hat{j} - x^2 y^2 \hat{k} \)

Solution
There really isn’t much to do here other than compute the divergence.

\[
\text{div } \vec{F} = \frac{\partial}{\partial x} \left( x^2 y \right) + \frac{\partial}{\partial y} \left( xyz \right) + \frac{\partial}{\partial z} \left( -x^2 y^2 \right) = 2xy + xz
\]

We also have the following fact about the relationship between the curl and the divergence.

\[
\text{div} \left( \text{curl } \vec{F} \right) = 0
\]
Example 3  Verify the above fact for the vector field \( \vec{F} = yz^2 \hat{i} + xy \hat{j} + yz \hat{k} \).

Solution
Let’s first compute the curl.

\[
\text{curl} \, \vec{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
yz^2 & xy & yz
\end{vmatrix}
\]

\[
= z \hat{i} + 2yz \hat{j} + y \hat{k} - z^2 \hat{k}
\]

\[
= z \hat{i} + 2yz \hat{j} + (y - z^2) \hat{k}
\]

Now compute the divergence of this.

\[
\text{div} \left( \text{curl} \, \vec{F} \right) = \frac{\partial}{\partial x} (z) + \frac{\partial}{\partial y} (2yz) + \frac{\partial}{\partial z} (y-z^2) = 2z - 2z = 0
\]

We also have a physical interpretation of the divergence. If we again think of \( \vec{F} \) as the velocity field of a flowing fluid then \( \text{div} \, \vec{F} \) represents the net rate of change of the mass of the fluid flowing from the point \((x, y, z)\) per unit volume. This can also be thought of as the tendency of a fluid to diverge from a point. If \( \text{div} \, \vec{F} = 0 \) then the \( \vec{F} \) is called incompressible.

The next topic that we want to briefly mention is the Laplace operator. Let’s first take a look at,

\[
\text{div} (\nabla f) = \nabla \cdot \nabla f = f_{xx} + f_{yy} + f_{zz}
\]

The Laplace operator is then defined as,

\[
\nabla^2 = \nabla \cdot \nabla
\]

The Laplace operator arises naturally in many fields including heat transfer and fluid flow.

The final topic in this section is to give two vector forms of Green’s Theorem. The first form uses the curl of the vector field and is,

\[
\oint_C \vec{F} \cdot d\vec{r} = \iint_D \left( \text{curl} \, \vec{F} \right) \cdot \hat{k} \, dA
\]

where \( \hat{k} \) is the standard unit vector in the positive \( z \) direction.

The second form uses the divergence. In this case we also need the outward unit normal to the curve \( C \). If the curve is parameterized by

\[
\vec{r}(t) = x(t) \hat{i} + y(t) \hat{j}
\]

then the outward unit normal is given by,

\[
\vec{n} = \frac{y'(t) \hat{i} - x'(t) \hat{j}}{\| \vec{r}'(t) \|}
\]

Here is a sketch illustrating the outward unit normal for some curve \( C \) at various points.
The vector form of Green’s Theorem that uses the divergence is given by,

\[ \int_C \vec{F} \cdot \hat{n} \, ds = \int_D \text{div} \, \vec{F} \, dA \]