Preface

Here are the solutions to the practice problems for my Calculus II notes. Some solutions will have more or less detail than other solutions. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you’ve reached the level of working the harder problems then you will probably already understand the basics fairly well and won’t need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven’t been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.
1. Compute \( \text{div} \vec{F} \) and \( \text{curl} \vec{F} \) for \( \vec{F} = x^2y\hat{i} - (z^3 - 3x)\hat{j} + 4y^2\hat{k} \).

Step 1
Let’s compute the divergence first and there isn’t much to do other than run through the formula.

\[
\text{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x} (x^2y) + \frac{\partial}{\partial y} (3x - z^3) + \frac{\partial}{\partial z} (4y^2) = 2xy
\]

Be careful to watch for minus signs in front of any of the vector components (2\textsuperscript{nd} component in this case!). It is easy to get in a hurry and miss them.

Step 2
The curl is a little more work but still just formula work so here is the curl.

\[
\text{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
x^2y & 3x - z^3 & 4y^2
\end{vmatrix}
\]

\[
= \frac{\partial}{\partial y} (4y^2)\hat{i} + \frac{\partial}{\partial z} (x^2y)\hat{j} - \frac{\partial}{\partial x} (x^2y)\hat{k} - \frac{\partial}{\partial y} (3x - z^3)\hat{i} + \frac{\partial}{\partial z} (4y^2)\hat{j} - \frac{\partial}{\partial x} (3x - z^3)\hat{k}
\]

\[
= 8y\hat{i} + 3\hat{k} - x^2\hat{k} + 3z^2\hat{i}
\]

\[
= (8y + 3z^2)\hat{i} + (3 - x^2)\hat{k}
\]

Again, don’t forget the minus sign on the 2\textsuperscript{nd} component.

2. Compute \( \text{div} \vec{F} \) and \( \text{curl} \vec{F} \) for \( \vec{F} = (3x + 2z^2)\hat{i} + \frac{x^3y^2}{z}\hat{j} - (z - 7x)\hat{k} \).

Step 1
Let’s compute the divergence first and there isn’t much to do other than run through the formula.

\[
\text{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x} (3x + 2z^2) + \frac{\partial}{\partial y} \left(\frac{x^3y^2}{z}\right) + \frac{\partial}{\partial z} (7x - z) = 2 + \frac{2x^3y}{z}
\]

Be careful to watch for minus signs in front of any of the vector components (3\textsuperscript{rd} component in this case!). It is easy to get in a hurry and miss them.
Step 2
The curl is a little more work but still just formula work so here is the curl.

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
3x + 2z^2 & \frac{x^3 y^2}{z} & 7x - z
\end{vmatrix}$$

$$= \frac{\partial}{\partial y} (7x - z) \vec{i} + \frac{\partial}{\partial z} (3x + 2z^2) \vec{j} + \frac{\partial}{\partial x} \left( \frac{x^3 y^2}{z} \right) \vec{k}$$

$$- \frac{\partial}{\partial y} (3x + 2z^2) \vec{k} - \frac{\partial}{\partial x} (7x - z) \vec{j} - \frac{\partial}{\partial z} \left( \frac{x^3 y^2}{z} \right) \vec{i}$$

$$= 4z \vec{j} + \frac{3x^2 y^2}{z} \vec{k} - 7 \vec{j} + \frac{x^3 y^2}{z^2} \vec{k}$$

$$= \frac{x^3 y^2}{z^2} \vec{k} - (4z - 7) \vec{j} + \frac{3x^2 y^2}{z} \vec{k}$$

Again, don’t forget the minus sign on the 3rd component.

3. Determine if the following vector field is conservative.

$$\vec{F} = \left(4y^2 + \frac{3x^2 y}{z^2} \right) \vec{i} + \left(8xy + \frac{x^3}{z^2} \right) \vec{j} + \left(11 - \frac{2x^3 y}{z^2} \right) \vec{k}$$

Step 1
We know all we need to do here is compute the curl of the vector field.
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curl $\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4y^2 + \frac{3x^2y}{z^2} & 8xy + \frac{x^3}{z^2} & 11 - \frac{2x^3y}{z^3} \end{vmatrix}

= \frac{\partial}{\partial y} \left( 8xy + \frac{x^3}{z^2} \right) \hat{k} - \frac{\partial}{\partial x} \left( 11 - \frac{2x^3y}{z^3} \right) \hat{j} - \frac{\partial}{\partial z} \left( 8xy + \frac{x^3}{z^2} \right) \hat{i}

= -\frac{2x^3}{z^3} \hat{i} - \frac{6x^2y}{z^3} \hat{j} + \left( \frac{8y + 3x^2}{z^2} \right) \hat{k} - \left( \frac{8y + 3x^2}{z^2} \right) \hat{k} + \frac{6x^3y}{z^3} \hat{j} + \frac{2x^3}{z^3} \hat{i}

= \vec{0}

Step 2
So, we found that $\text{curl } \vec{F} = \vec{0}$ for this vector field and so the vector field is conservative.

4. Determine if the following vector field is conservative.

$\vec{F} = 6x \hat{i} + \left( 2y - y^2 \right) \hat{j} + \left( 6z - x^3 \right) \hat{k}$

Step 1
We know all we need to do here is compute the curl of the vector field.

curl $\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6x & 2y - y^2 & 6z - x^3 \end{vmatrix}

= \frac{\partial}{\partial y} (6z - x^3) \hat{i} + \frac{\partial}{\partial z} (6x) \hat{j} + \frac{\partial}{\partial x} (2y - y^2) \hat{k}

= 3x^2 \hat{j}

Step 2
So, we found that $\text{curl } \vec{F} = 3x^2 \hat{j} \neq \vec{0}$ for this vector field and so the vector field is NOT conservative.