Preface

Here are my online notes for my Calculus III course that I teach here at Lamar University. Despite the fact that these are my “class notes” they should be accessible to anyone wanting to learn Calculus III or needing a refresher in some of the topics from the class.

These notes do assume that the reader has a good working knowledge of Calculus I topics including limits, derivatives and integration. It also assumes that the reader has a good knowledge of several Calculus II topics including some integration techniques, parametric equations, vectors, and knowledge of three dimensional space.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Calculus III many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don’t have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren’t worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
Equations of Planes

In the first section of this chapter we saw a couple of equations of planes. However, none of those equations had three variables in them and were really extensions of graphs that we could look at in two dimensions. We would like a more general equation for planes.

So, let’s start by assuming that we know a point that is on the plane, \( P_0 = (x_0, y_0, z_0) \). Let’s also suppose that we have a vector that is orthogonal (perpendicular) to the plane, \( \vec{n} = (a, b, c) \). This vector is called the normal vector. Now, assume that \( P = (x, y, z) \) is any point in the plane.

Finally, since we are going to be working with vectors initially we’ll let \( \vec{r}_0 \) and \( \vec{r} \) be the position vectors for \( P_0 \) and \( P \) respectively.

Here is a sketch of all these vectors.

Notice that we added in the vector \( \vec{r} - \vec{r}_0 \) which will lie completely in the plane. Also notice that we put the normal vector on the plane, but there is actually no reason to expect this to be the case. We put it here to illustrate the point. It is completely possible that the normal vector does not touch the plane in any way.

Now, because \( \vec{n} \) is orthogonal to the plane, it’s also orthogonal to any vector that lies in the plane. In particular it’s orthogonal to \( \vec{r} - \vec{r}_0 \). Recall from the Dot Product section that two orthogonal vectors will have a dot product of zero. In other words,

\[
\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \quad \Rightarrow \quad \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0
\]

This is called the vector equation of the plane.
A slightly more useful form of the equations is as follows. Start with the first form of the vector equation and write down a vector for the difference.

\[ \langle a, b, c \rangle \left( \langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle \right) = 0 \]

\[ \langle a, b, c \rangle \times \langle x - x_0, y - y_0, z - z_0 \rangle = 0 \]

Now, actually compute the dot product to get,

\[ a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \]

This is called the **scalar equation of plane**. Often this will be written as,

\[ ax + by + cz = d \]

where \( d = ax_0 + by_0 + cz_0 \).

This second form is often how we are given equations of planes. Notice that if we are given the equation of a plane in this form we can quickly get a normal vector for the plane. A normal vector is,

\[ \vec{n} = \langle a, b, c \rangle \]

Let's work a couple of examples.

**Example 1** Determine the equation of the plane that contains the points \( P = (1, -2, 0) \), \( Q = (3, 1, 4) \) and \( R = (0, -1, 2) \).

**Solution**

In order to write down the equation of plane we need a point (we’ve got three so we’re cool there) and a normal vector. We need to find a normal vector. Recall however, that we saw how to do this in the [Cross Product] section.

We can form the following two vectors from the given points.

\[ \overrightarrow{PQ} = \langle 2, 3, 4 \rangle \quad \overrightarrow{PR} = \langle -1, 1, 2 \rangle \]

These two vectors will lie completely in the plane since we formed them from points that were in the plane. Notice as well that there are many possible vectors to use here, we just chose two of the possibilities.

Now, we know that the cross product of two vectors will be orthogonal to both of these vectors. Since both of these are in the plane any vector that is orthogonal to both of these will also be orthogonal to the plane. Therefore, we can use the cross product as the normal vector.

\[
\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 4 \\ -1 & 1 & 2 \end{vmatrix} = 2\vec{i} - 8\vec{j} + 5\vec{k}
\]

The equation of the plane is then,
Example 2 Determine if the plane given by \(-x + 2z = 10\) and the line given by 
\[ \vec{r} = \langle 5, 2-t, 10+4t \rangle \]
are orthogonal, parallel or neither.

Solution
This is not as difficult a problem as it may at first appear to be. We can pick off a vector that is 
normal to the plane. This is \(\vec{n} = \langle -1, 0, 2 \rangle\). We can also get a vector that is parallel to the line. 
This is \(\vec{v} = \langle 0, -1, 4 \rangle\).

Now, if these two vectors are parallel then the line and the plane will be orthogonal. If you think 
about it this makes some sense. If \(\vec{n}\) and \(\vec{v}\) are parallel, then \(\vec{v}\) is orthogonal to the plane, but \(\vec{v}\) 
is also parallel to the line. So, if the two vectors are parallel the line and plane will be orthogonal.

Let’s check this.

\[ \vec{n} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 2 \\ 0 & -1 & 4 \end{vmatrix} = \vec{i}(-2) - \vec{j}(-1) + \vec{k}(0) = \vec{-2i} + \vec{j} + \vec{0} \neq \vec{0} \]

So, the vectors aren’t parallel and so the plane and the line are not orthogonal.

Now, let’s check to see if the plane and line are parallel. If the line is parallel to the plane then 
any vector parallel to the line will be orthogonal to the normal vector of the plane. In other 
words, if \(\vec{n}\) and \(\vec{v}\) are orthogonal then the line and the plane will be parallel.

Let’s check this. 
\[ \vec{n} \cdot \vec{v} = 0 \cdot -1 + \vec{0} + 8 = 8 \neq 0 \]

The two vectors aren’t orthogonal and so the line and plane aren’t parallel.

So, the line and the plane are neither orthogonal nor parallel.