Preface

Here are my online notes for my Calculus III course that I teach here at Lamar University. Despite the fact that these are my “class notes” they should be accessible to anyone wanting to learn Calculus III or needing a refresher in some of the topics from the class.

These notes do assume that the reader has a good working knowledge of Calculus I topics including limits, derivatives and integration. It also assumes that the reader has a good knowledge of several Calculus II topics including some integration techniques, parametric equations, vectors, and knowledge of three dimensional space.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Calculus III many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don’t have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren’t worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
Curvature

In this section we want to briefly discuss the curvature of a smooth curve (recall that for a smooth curve we require \( \vec{r}'(t) \) is continuous and \( \vec{r}'(t) \neq 0 \)). The curvature measures how fast a curve is changing direction at a given point.

There are several formulas for determining the curvature for a curve. The formal definition of curvature is,

\[
\kappa = \frac{dT}{ds}
\]

where \( T \) is the unit tangent and \( s \) is the arc length. Recall that we saw in a previous section how to reparameterize a curve to get it into terms of the arc length.

In general the formal definition of the curvature is not easy to use so there are two alternate formulas that we can use. Here they are.

\[
\kappa = \frac{\|r''(t)\times r''(t)\|}{\|r'(t)\|^3}
\]

These may not be particularly easy to deal with either, but at least we don’t need to reparameterize the unit tangent.

**Example 1** Determine the curvature for \( \vec{r}(t) = (t, 3\sin t, 3\cos t) \).

**Solution**

Back in the section when we introduced the tangent vector we computed the tangent and unit tangent vectors for this function. These were,

\[
\vec{r}'(t) = \langle 1, 3\cos t, -3\sin t \rangle
\]

\[
\vec{T}(t) = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\cos t, -\frac{3}{\sqrt{10}}\sin t \right\rangle
\]

The derivative of the unit tangent is,

\[
\vec{T}'(t) = \left\langle 0, -\frac{3}{\sqrt{10}}\sin t, -\frac{3}{\sqrt{10}}\cos t \right\rangle
\]

The magnitudes of the two vectors are,

\[
\|\vec{r}''(t)\| = \sqrt{1 + 9\cos^2 t + 9\sin^2 t} = \sqrt{10}
\]

\[
\|\vec{T}'(t)\| = \sqrt{0 + \frac{9}{10}\sin^2 t + \frac{9}{10}\cos^2 t} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}
\]

The curvature is then,
In this case the curvature is constant. This means that the curve is changing direction at the same rate at every point along it. Recalling that this curve is a helix this result makes sense.

**Example 2** Determine the curvature of \( \vec{r}(t) = t^2 \vec{i} + t \vec{k} \).

**Solution**
In this case the second form of the curvature would probably be easiest. Here are the first couple of derivatives.

\[
\vec{r}'(t) = 2t \vec{i} + \vec{k} \quad \vec{r}''(t) = 2 \vec{i}
\]

Next, we need the cross product.

\[
\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
2t & 0 & 1 \\
2 & 0 & 0
\end{vmatrix} = 2 \vec{j}
\]

The magnitudes are,

\[
\|\vec{r}'(t) \times \vec{r}''(t)\| = 2 \quad \|\vec{r}'(t)\| = \sqrt{4t^2 + 1}
\]

The curvature at any value of \( t \) is then,

\[
\kappa = \frac{2}{(4t^2 + 1)^{3/2}}
\]

There is a special case that we can look at here as well. Suppose that we have a curve given by \( y = f(x) \) and we want to find its curvature.

As we saw when we first looked at vector functions we can write this as follows,

\[
\vec{r}(x) = x \vec{i} + f(x) \vec{j}
\]

If we then use the second formula for the curvature we will arrive at the following formula for the curvature.

\[
\kappa = \frac{|f''(x)|}{\left(1 + [f'(x)]^2\right)^{3/2}}
\]