Preface

Here are my online notes for my Calculus I course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Calculus I or needing a refresher in some of the early topics in calculus.

I’ve tried to make these notes as self contained as possible and so all the information needed to read through them is either from an Algebra or Trig class or contained in other sections of the notes.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. Because I want these notes to provide some more examples for you to read through, I don’t always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than are presented here.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible when writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
Limits At Infinity, Part I

In the previous section we saw limits that were infinity and it’s now time to take a look at limits at infinity. By limits at infinity we mean one of the following two limits.

\[ \lim_{x \to \infty} f(x) \quad \text{or} \quad \lim_{x \to -\infty} f(x) \]

In other words, we are going to be looking at what happens to a function if we let \( x \) get very large in either the positive or negative sense. Also, as we’ll soon see, these limits may also have infinity as a value.

First, let’s note that the set of Facts from the Infinite Limit section also hold if the replace the \( \lim_{x \to c} \) with \( \lim_{x \to \infty} \) or \( \lim_{x \to -\infty} \). The proof of this is nearly identical to the proof of the original set of facts with only minor modifications to handle the change in the limit and so is left to the reader. We won’t need these facts much over the next couple of sections but they will be required on occasion.

In fact, many of the limits that we’re going to be looking at we will need the following two facts.

Fact 1

1. If \( r \) is a positive rational number and \( c \) is any real number then,

\[ \lim_{x \to \infty} \frac{c}{x} = 0 \]

2. If \( r \) is a positive rational number, \( c \) is any real number and \( x' \) is defined for \( x < 0 \) then,

\[ \lim_{x \to -\infty} \frac{c}{x'} = 0 \]

The first part of this fact should make sense if you think about it. Because we are requiring \( r > 0 \) we know that \( x' \) will stay in the denominator. Next as we increase \( x \) then \( x' \) will also increase. So, we have a constant divided by an increasingly large number and so the result will be increasingly small. Or, in the limit we will get zero.

The second part is nearly identical except we need to worry about \( x' \) being defined for negative \( x \). This condition is here to avoid cases such as \( r = \frac{1}{2} \). If this \( r \) were allowed then we’d be taking the square root of negative numbers which would be complex and we want to avoid that at this level.

Note as well that the sign of \( c \) will not affect the answer. Regardless of the sign of \( c \) we’ll still have a constant divided by a very large number which will result in a very small number and the larger \( x \) get the smaller the fraction gets. The sign of \( c \) will affect which direction the fraction approaches zero (i.e. from the positive or negative side) but it still approaches zero.
If you think about it this is really a special case of the last Fact from the Facts in the previous section. However, to see a direct proof of this fact see the Proof of Various Limit Properties section in the Extras chapter.

Let’s start the off the examples with one that will lead us to a nice idea that we’ll use on a regular basis about limits at infinity for polynomials.

**Example 1** Evaluate each of the following limits.

(a) \( \lim_{x \to \infty} (2x^4 - x^2 - 8x) \) [Solution]

(b) \( \lim_{t \to -\infty} \left( \frac{1}{3}t^5 + 2t^3 - t^2 + 8 \right) \) [Solution]

**Solution**

(a) \( \lim_{x \to \infty} (2x^4 - x^2 - 8x) \)

Our first thought here is probably to just “plug” infinity into the polynomial and “evaluate” each term to determine the value of the limit. It is pretty simple to see what each term will do in the limit and so this seems like an obvious step, especially since we’ve been doing that for other limits in previous sections.

So, let’s see what we get if we do that. As \( x \) approaches infinity, then \( x \) to a power can only get larger and the coefficient on each term (the first and third) will only make the term even larger. So, if we look at what each term is doing in the limit we get the following,

\[
\lim_{x \to \infty} (2x^4 - x^2 - 8x) = \infty - \infty - \infty
\]

Now, we’ve got a small, but easily fixed, problem to deal with. We are probably tempted to say that the answer is zero (because we have an infinity minus an infinity) or maybe \(-\infty \) (because we’re subtracting two infinities off of one infinity). However, in both cases we’d be wrong. This is one of those indeterminate forms that we first started seeing in a previous section.

Infinities just don’t always behave as real numbers do when it comes to arithmetic. Without more work there is simply no way to know what \( \infty - \infty \) will be and so we really need to be careful with this kind of problem. To read a little more about this see the Types of Infinity section in the Extras chapter.

So, we need a way to get around this problem. What we’ll do here is factor the largest power of \( x \) out of the whole polynomial as follows,

\[
\lim_{x \to \infty} (2x^4 - x^2 - 8x) = \lim_{x \to \infty} \left[ x^4 \left( 2 - \frac{1}{x^2} - \frac{8}{x^4} \right) \right]
\]

If you’re not sure you agree with the factoring above (there’s a chance you haven’t really been asked to do this kind of factoring prior to this) then recall that to check all you need to do is
multiply the \(x^4\) back through the parenthesis to verify it was done correctly. Also, an easy way to remember how to do this kind of factoring is to note that the second term is just the original polynomial divided by \(x^4\). This will always work when factoring a power of \(x\) out of a polynomial.

Now for each of the terms we have,

\[
\lim_{x \to \infty} x^4 = \infty \quad \text{and} \quad \lim_{x \to \infty} \left(2 - \frac{1}{x^2} - \frac{8}{x^3}\right) = 2
\]

The first limit is clearly infinity and for the second limit we’ll use the fact above on the last two terms. Therefore using Fact 2 from the previous section we see value of the limit will be,

\[
\lim_{x \to \infty} (2x^4 - x^2 - 8x) = \infty
\]

(b) \(\lim_{t \to -\infty} \left(\frac{1}{3} t^5 + 2t^3 - t^2 + 8\right)\)

We’ll work this part much quicker than the previous part. All we need to do is factor out the largest power of \(t\) to get the following,

\[
\lim_{t \to -\infty} \left(\frac{1}{3} t^5 + 2t^3 - t^2 + 8\right) = \lim_{t \to -\infty} \left[t^2 \left(\frac{1}{3} \frac{1}{t^3} + \frac{2}{t^2} - \frac{1}{t} + \frac{8}{t^5}\right)\right]
\]

Remember that all you need to do to get the factoring correct is divide the original polynomial by the power of \(t\) we’re factoring out, \(t^5\) in this case.

Now all we need to do is take the limit of the two terms. In the first don’t forget that since we’re going out towards \(-\infty\) and we’re raising \(t\) to the 5th power that the limit will be negative (negative number raised to an odd power is still negative). In the second term we’ll again make heavy use of the fact above to see that is a finite number.

Therefore, using the a modification of the Facts from the previous section the value of the limit is,

\[
\lim_{t \to -\infty} \left(\frac{1}{3} t^5 + 2t^3 - t^2 + 8\right) = -\infty
\]

Okay, now that we’ve seen how a couple of polynomials work we can give a simple fact about polynomials in general.

**Fact 2**

If \(p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0\) is a polynomial of degree \(n\) (i.e. \(a_n \neq 0\)) then,
\[
\lim_{x \to \infty} p(x) = \lim_{x \to \infty} a_n x^n
\]

What this fact is really saying is that when we go to take a limit at infinity for a polynomial then all we need to really do is look at the term with the largest power and ask what that term is doing in the limit since the polynomial will have the same behavior.

You can see the proof in the Proof of Various Limit Properties section in the Extras chapter.

Let’s now move into some more complicated limits.

**Example 2** Evaluate both of the following limits.

\[
\lim_{x \to \infty} \frac{2x^4 - x^2 + 8x}{-5x^4 + 7}, \quad \lim_{x \to \infty} \frac{2x^4 - x^2 + 8x}{-5x^4 + 7}
\]

**Solution**

First, the only difference between these two is that one is going to positive infinity and the other is going to negative infinity. Sometimes this small difference will affect then value of the limit and at other times it won’t.

Let’s start with the first limit and as with our first set of examples it might be tempting to just “plug” in the infinity. Since both the numerator and denominator are polynomials we can use the above fact to determine the behavior of each. Doing this gives,

\[
\lim_{x \to \infty} \frac{2x^4 - x^2 + 8x}{-5x^4 + 7} = \frac{\infty}{-\infty}
\]

This is yet another indeterminate form. In this case we might be tempted to say that the limit is infinity (because of the infinity in the numerator), zero (because of the infinity in the denominator) or -1 (because something divided by itself is one). There are three separate arithmetic “rules” at work here and without work there is no way to know which “rule” will be correct and to make matters worse it’s possible that none of them may work and we might get a completely different answer, say \(-\frac{2}{5}\) to pick a number completely at random.

So, when we have a polynomial divided by a polynomial we’re going to proceed much as we did with only polynomials. We first identify the largest power of \(x\) in the denominator (and yes, we only look at the denominator for this) and we then factor this out of both the numerator and denominator. Doing this for the first limit gives,

\[
\lim_{x \to \infty} \frac{2x^4 - x^2 + 8x}{-5x^4 + 7} = \lim_{x \to \infty} \frac{2 - \frac{1}{x^2} + \frac{8}{x^3}}{-5 + \frac{7}{x^4}}
\]

Once we’ve done this we can cancel the \(x^4\) from both the numerator and the denominator and
then use the Fact 1 above to take the limit of all the remaining terms. This gives,

\[
\lim_{x \to \infty} \frac{2x^4 - x^2 + 8x}{-5x^4 + 7} \rightarrow \lim_{x \to \infty} \frac{2 - \frac{1}{x^2} + \frac{8}{x^3}}{-5 + \frac{7}{x^4}}
\]

\[
= \frac{2 + 0 + 0}{-5 + 0} = \frac{2}{-5} = -\frac{2}{5}
\]

In this case the indeterminate form was neither of the “obvious” choices of infinity, zero, or -1 so be careful with make these kinds of assumptions with this kind of indeterminate forms.

The second limit is done in a similar fashion. Notice however, that nowhere in the work for the first limit did we actually use the fact that the limit was going to plus infinity. In this case it doesn’t matter which infinity we are going towards we will get the same value for the limit.

\[
\lim_{x \to \infty} \frac{2x^4 - x^2 + 8x}{-5x^4 + 7} = \frac{2}{-5} = -\frac{2}{5}
\]

In the previous example the infinity that we were using in the limit didn’t change the answer. This will not always be the case so don’t make the assumption that this will always be the case.

Let’s take a look at an example where we get different answers for each limit.

**Example 3** Evaluate each of the following limits.

\[
\lim_{x \to \infty} \frac{\sqrt{3x^2 + 6}}{5 - 2x} \quad \text{and} \quad \lim_{x \to \infty} \frac{\sqrt{3x^2 + 6}}{5 - 2x}
\]

**Solution**

The square root in this problem won’t change our work, but it will make the work a little messier.

Let’s start with the first limit. In this case the largest power of \(x\) in the denominator is just an \(x\). So we need to factor an \(x\) out of the numerator and the denominator. When we are done factoring the \(x\) out we will need an \(x\) in both of the numerator and the denominator. To get this in the numerator we will have to factor an \(x^2\) out of the square root so that after we take the square root we will get an \(x\).

This is probably not something you’re used to doing, but just remember that when it comes out of the square root it needs to be an \(x\) and the only way have an \(x\) come out of a square is to take the square root of \(x^2\) and so that is what we’ll need to factor out of the term under the radical. Here’s the factoring work for this part,
\[
\lim_{x \to \infty} \frac{\sqrt{3x^2 + 6}}{5 - 2x} = \lim_{x \to \infty} \frac{\sqrt{x^2 \left( \frac{3 + \frac{6}{x^2}}{x - \frac{2}{x}} \right)}}{\frac{5}{x} - 2} = \lim_{x \to \infty} \frac{\sqrt{x^2 \left( \frac{3 + \frac{6}{x^2}}{x - \frac{2}{x}} \right)}}{\frac{5}{x} - 2}
\]

This is where we need to be really careful with the square root in the problem. Don’t forget that \( \sqrt{x^2} = |x| \)

Square roots are ALWAYS positive and so we need the absolute value bars on the \( x \) to make sure that it will give a positive answer. This is not something that most people ever remember seeing in an Algebra class and in fact it’s not always given in an Algebra class. However, at this point it becomes absolutely vital that we know and use this fact. Using this fact the limit becomes,

\[
\lim_{x \to \infty} \frac{\sqrt{3x^2 + 6}}{5 - 2x} = \lim_{x \to \infty} \frac{|x| \sqrt{\frac{3 + \frac{6}{x^2}}{\frac{5}{x} - 2}}}{\frac{5}{x} - 2}
\]

Now, we can’t just cancel the \( x \)'s. We first will need to get rid of the absolute value bars. To do this let’s recall the definition of absolute value.

\[
|x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0 
\end{cases}
\]

In this case we are going out to plus infinity so we can safely assume that the \( x \) will be positive and so we can just drop the absolute value bars. The limit is then,

\[
\lim_{x \to \infty} \frac{\sqrt{3x^2 + 6}}{5 - 2x} = \lim_{x \to \infty} \frac{x \sqrt{\frac{3 + \frac{6}{x^2}}{\frac{5}{x} - 2}}}{\frac{5}{x} - 2} = \lim_{x \to \infty} \frac{\sqrt{\frac{3 + \frac{6}{x^2}}{\frac{5}{x} - 2}}}{{\frac{5}{x} - 2}} = \frac{\sqrt{3} + 0}{0 - 2} = -\frac{\sqrt{3}}{2}
\]

Let’s now take a look at the second limit (the one with negative infinity). In this case we will need to pay attention to the limit that we are using. The initial work will be the same up until we reach the following step.
In this limit we are going to minus infinity so in this case we can assume that \( x \) is negative. So, in order to drop the absolute value bars in this case we will need to tack on a minus sign as well. The limit is then,

\[
\lim_{x \to -\infty} \frac{\sqrt{3x^2 + 6}}{5 - 2x} = \lim_{x \to -\infty} \frac{|x|\sqrt{\frac{3 + 6}{x^2}}}{x\left(\frac{5}{x} - 2\right)}
\]

So, as we saw in the last two examples sometimes the infinity in the limit will affect the answer and other times it won’t. Note as well that it doesn’t always just change the sign of the number. It can on occasion completely change the value. We’ll see an example of this later in this section.

Before moving on to a couple of more examples let’s revisit the idea of asymptotes that we first saw in the previous section. Just as we can have vertical asymptotes defined in terms of limits we can also have horizontal asymptotes defined in terms of limits.

**Definition**

The function \( f(x) \) will have a horizontal asymptote at \( y = L \) if either of the following are true.

\[
\lim_{x \to \infty} f(x) = L \quad \lim_{x \to -\infty} f(x) = L
\]

We’re not going to be doing much with asymptotes here, but it’s an easy fact to give and we can use the previous example to illustrate all the asymptote ideas we’ve seen in the both this section and the previous section. The function in the last example will have two horizontal asymptotes. It will also have a vertical asymptote. Here is a graph of the function showing these.
Let's work another couple of examples involving rational expressions.

**Example 4** Evaluate each of the following limits.

\[
\lim_{z \to \infty} \frac{4z^2 + z^6}{1 - 5z^3} \quad \text{and} \quad \lim_{z \to -\infty} \frac{4z^2 + z^6}{1 - 5z^3}
\]

**Solution**

Let's do the first limit and in this case it looks like we will factor a \(z^3\) out of both the numerator and denominator. Remember that we only look at the denominator when determining the largest power of \(z\) here. There is a larger power of \(z\) in the numerator but we ignore it. We ONLY look at the denominator when doing this! So doing the factoring gives,

\[
\lim_{z \to \infty} \frac{4z^2 + z^6}{1 - 5z^3} = \lim_{z \to \infty} \frac{z^3 \left( \frac{4}{z} + z^3 \right)}{z^3 \left( \frac{1}{z^3} - 5 \right)}
\]

\[
= \lim_{z \to \infty} \frac{\frac{4}{z} + z^3}{\frac{1}{z^3} - 5}
\]

When we take the limit we'll need to be a little careful. The first term in the numerator and denominator will both be zero. However, the \(z^3\) in the numerator will be going to plus infinity in the limit and so the limit is,

\[
\lim_{z \to \infty} \frac{4z^2 + z^6}{1 - 5z^3} = \frac{\infty}{-5} = -\infty
\]

The final limit is negative because we have a quotient of positive quantity and a negative quantity.

Now, let's take a look at the second limit. Note that the only different in the work is at the final “evaluation” step and so we'll pick up the work there.
In this case the \( z^3 \) in the numerator gives negative infinity in the limit since we are going out to minus infinity and the power is odd. The answer is positive since we have a quotient of two negative numbers.

Example 5  Evaluate the following limit.

\[
\lim_{t \to -\infty} \frac{t^2 - 5t - 9}{2t^4 + 3t^3}
\]

Solution

In this case it looks like we will factor a \( t^4 \) out of both the numerator and denominator. Doing this gives,

\[
\lim_{t \to -\infty} \frac{t^2 - 5t - 9}{2t^4 + 3t^3} = \lim_{t \to -\infty} \frac{t^4 \left( \frac{1}{t^2} - \frac{5}{t^3} - \frac{9}{t^4} \right)}{t^4 \left( 2 + \frac{3}{t} \right)}
\]

\[
= \lim_{t \to -\infty} \frac{1}{2 + \frac{3}{t}} \left( \frac{1}{t^2} - \frac{5}{t^3} - \frac{9}{t^4} \right)
\]

\[
= \frac{0}{2} = 0
\]

In this case using Fact 1 we can see that the numerator is zero and so since the denominator is also not zero the fraction, and hence the limit, will be zero.

In this section we concentrated on limits at infinity with functions that only involved polynomials and/or rational expression involving polynomials. There are many more types of functions that we could use here. That is the subject of the next section.

To see a precise and mathematical definition of this kind of limit see the The Definition of the Limit section at the end of this chapter.