Here are my online notes for my Calculus I course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Calculus I or needing a refresher in some of the early topics in calculus.

I’ve tried to make these notes as self contained as possible and so all the information needed to read through them is either from an Algebra or Trig class or contained in other sections of the notes.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. Because I want these notes to provide some more examples for you to read through, I don’t always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than are presented here.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible when writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
**Limits At Infinity, Part II**

In the previous section we looked at limits at infinity of polynomials and/or rational expression involving polynomials. In this section we want to take a look at some other types of functions that often show up in limits at infinity. The functions we’ll be looking at here are exponentials, natural logarithms and inverse tangents.

Let’s start by taking a look at some of very basic examples involving exponential functions.

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**Example 1** Evaluate each of the following limits.

\[
\lim_{x \to \infty} e^x \quad \lim_{x \to -\infty} e^x \quad \lim_{x \to \infty} e^{-x} \quad \lim_{x \to -\infty} e^{-x}
\]

**Solution**

There are really just restatements of facts given in the [basic exponential section](#) of the review so we’ll leave it to you to go back and verify these.

\[
\lim_{x \to \infty} e^x = \infty \quad \lim_{x \to -\infty} e^x = 0 \quad \lim_{x \to \infty} e^{-x} = 0 \quad \lim_{x \to -\infty} e^{-x} = \infty
\]

The main point of this example was to point out that if the exponent of an exponential goes to infinity in the limit then the exponential function will also go to infinity in the limit. Likewise, if the exponent goes to minus infinity in the limit then the exponential will go to zero in the limit.

Here’s a quick set of examples to illustrate these ideas.

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**Example 2** Evaluate each of the following limits.

(a) \( \lim_{x \to \infty} e^{2-4x-8x^2} \)  [Solution]

(b) \( \lim_{t \to -\infty} e^{t^4-5t^2+1} \)  [Solution]

(c) \( \lim_{z \to 0^+} e^{\frac{1}{z}} \)  [Solution]

**Solution**

(a) \( \lim_{x \to \infty} e^{2-4x-8x^2} \)

In this part what we need to note (using Fact 2 above) is that in the limit the exponent of the exponential does the following,

\[
\lim_{x \to \infty} \left(2 - 4x - 8x^2\right) = -\infty
\]

So, the exponent goes to minus infinity in the limit and so the exponential must go to zero in the limit using the ideas from the previous set of examples. So, the answer here is,

\[
\lim_{x \to \infty} e^{2-4x-8x^2} = 0
\]

[Return to Problems]
Here’s the first note that,  
\[
\lim_{t \to \infty} (t^4 - 5t^2 + 1) = \infty
\]

The exponent goes to infinity in the limit and so the exponential will also need to go to infinity in the limit. Or,  
\[
\lim_{t \to \infty} e^{t^4 - 5t^2 + 1} = \infty
\]

On the surface this part doesn’t appear to belong in this section since it isn’t a limit at infinity. However, it does fit into the ideas we’re examining in this set of examples.

So, let’s first note that using the idea from the previous section we have,  
\[
\lim_{z \to 0^+} \frac{1}{z} = \infty
\]

Remember that in order to do this limit here we do need to do a right hand limit.

So, the exponent goes to infinity in the limit and so the exponential must also go to infinity.

Here’s the answer to this part.  
\[
\lim_{z \to 0^+} e^{\frac{1}{z}} = \infty
\]

Let’s work some more complicated examples involving exponentials. In the following set of examples it won’t be that the exponents are more complicated, but instead that there will be more than one exponential function to deal with.

**Example 3** Evaluate each of the following limits.

(a)  
\[
\lim_{x \to \pm} \left( e^{10x} - 4e^{6x} + 3e^{x} + 2e^{-2x} - 9e^{-15x} \right)
\]

(b)  
\[
\lim_{x \to \pm} \left( e^{10x} - 4e^{6x} + 3e^{x} + 2e^{-2x} - 9e^{-15x} \right)
\]

**Solution**

So, the only difference between these two limits is the fact that in the first we’re taking the limit as we go to plus infinity and in the second we’re going to minus infinity. To this point we’ve been able to “reuse” work from the first limit in the at least a portion of the second limit. With exponentials that will often not be the case, we’re going to treat each of these as separate problems.
(a) \( \lim_{x \to \infty} \left( e^{10x} - 4e^{6x} + 3e^{x} + 2e^{-2x} - 9e^{-15x} \right) \)

Let’s start by just taking the limit of each of the pieces and see what we get.

\[
\lim_{x \to \infty} \left( e^{10x} - 4e^{6x} + 3e^{x} + 2e^{-2x} - 9e^{-15x} \right) = \infty - \infty + \infty + 0 - 0
\]

The last two terms aren’t any problem (they will be in the next part however, do you see that?). The first three are a problem however as they present us with another indeterminate form.

When dealing with polynomials we factored out the term with the largest exponent in it. Let’s do the same thing here. However, we now have to deal with both positive and negative exponents and just what do we mean by the “largest” exponent. When dealing with these here we look at the terms that are causing the problems and ask which is the largest exponent in those terms. So, since only the first three terms are causing us problems (i.e. they all evaluate to an infinity in the limit) we’ll look only at those.

So, since \(10x\) is the largest of the three exponents there we’ll “factor” an \( e^{10x} \) out of the whole thing. Just as with polynomials we do the factoring by, in essence, dividing each term by \( e^{10x} \) and remembering that to simply the division all we need to do is subtract the exponents. For example, let’s just take a look at the last term,

\[
\frac{-9e^{-15x}}{e^{10x}} = -9e^{-15x-10x} = -9e^{-25x}
\]

Doing factoring on all terms then gives,

\[
\lim_{x \to \infty} \left( e^{10x} - 4e^{6x} + 3e^{x} + 2e^{-2x} - 9e^{-15x} \right) = \lim_{x \to \infty} \left[ e^{10x} \left( 1 - 4e^{-4x} + 3e^{-9x} + 2e^{-12x} - 9e^{-25x} \right) \right]
\]

Notice that in doing this factoring all the remaining exponentials now have negative exponents and we know that for this limit (i.e. going out to positive infinity) these will all be zero in the limit and so will no longer cause problems.

We can now take the limit of the two factors. The first is clearly infinity and the second is clearly a finite number (one in this case) and so the Facts from the Infinite Limits section gives us the following limit,

\[
\lim_{x \to \infty} \left( e^{10x} - 4e^{6x} + 3e^{x} + 2e^{-2x} - 9e^{-15x} \right) = \infty
\]

To simplify the work here a little all we really needed to do was factor the \( e^{10x} \) out of the “problem” terms (the first three in this case) as follows,

\[
\lim_{x \to \infty} \left( e^{10x} - 4e^{6x} + 3e^{x} \right) = \lim_{x \to \infty} \left[ e^{10x} \left( 1 - 4e^{-4x} + 3e^{-9x} + 2e^{-12x} - 9e^{-25x} \right) \right]
\]

\[= (\infty)(1) + 0 - 0\]

\[= \infty\]
We factored the $e^{10x}$ out of all terms for the practice of doing the factoring and to avoid any issues with having the extra terms at the end. Note as well that while we wrote $(\infty)(1)$ for the limit of the first term we are really using the Facts from the Infinite Limit section to do that limit.

\[(b) \lim_{x \to -\infty} \left( e^{10x} - 4e^{6x} + 3e^{x} + 2e^{-2x} - 9e^{-15x} \right) \]

Let’s start this one off in the same manner as the first part. Let’s take the limit of each of the pieces. This time note that because our limit is going to negative infinity the first three exponentials will in fact go to zero (because their exponents go to minus infinity in the limit). The final two exponentials will go to infinity in the limit (because their exponents go to plus infinity in the limit).

Taking the limits gives,

\[
\lim_{x \to -\infty} \left( e^{10x} - 4e^{6x} + 3e^{x} + 2e^{-2x} - 9e^{-15x} \right) = 0 - 0 + \infty - \infty
\]

So, the last two terms are the problem here as they once again leave us with an indeterminate form. As with the first example we’re going to factor out the “largest” exponent in the last two terms. This time however, “largest” doesn’t refer to the bigger of the two numbers (-2 is bigger than -15). Instead we’re going to use “largest” to refer to the exponent that is farther away from zero. Using this definition of “largest” means that we’re going to factor an $e^{-15x}$ out.

Again, remember that to factor this out all we really are doing is dividing each term by $e^{-15x}$ and then subtracting exponents. Here’s the work for the first term as an example,

\[
\frac{e^{10x}}{e^{-15x}} = e^{10x-(-15x)} = e^{25x}
\]

As with the first part we can either factor it out of only the “problem” terms (i.e. the last two terms), or all the terms. For the practice we’ll factor it out of all the terms. Here is the factoring work for this limit,

\[
\lim_{x \to -\infty} \left( e^{10x} - 4e^{6x} + 3e^{x} + 2e^{-2x} - 9e^{-15x} \right) = \lim_{x \to -\infty} \left[ e^{25x} \left( e^{25x} - 4e^{21x} + 3e^{16x} + 2e^{13x} - 9 \right) \right]
\]

Finally, after taking the limit of the two terms (the first is infinity and the second is a negative, finite number) and recalling the Facts from the Infinite Limit section we see that the limit is,

\[
\lim_{x \to -\infty} \left( e^{10x} - 4e^{6x} + 3e^{x} + 2e^{-2x} - 9e^{-15x} \right) = -\infty
\]

So, when dealing with sums and/or differences of exponential functions we look for the exponential with the “largest” exponent and remember here that “largest” means the exponent farthest from zero. Also remember that if we’re looking at a limit at plus infinity only the
exponentials with positive exponents are going to cause problems so those are the only terms we look at in determining the largest exponent. Likewise, if we are looking at a limit at minus infinity then only exponentials with negative exponents are going to cause problems and so only those are looked at in determining the largest exponent.

Finally, as you might have been able to guess from the previous example when dealing with a sum and/or difference of exponentials all we need to do is look at the largest exponent to determine the behavior of the whole expression. Again, remembering that if the limit is at plus infinity we only look at exponentials with positive exponents and if we’re looking at a limit at minus infinity we only look at exponentials with negative exponents.

Let’s next take a look at some rational functions involving exponentials.

**Example 4** Evaluate each of the following limits.

(a) \[ \lim_{{x \to \infty}} \frac{6e^{4x} - e^{2x}}{8e^{4x} - e^{2x} + 3e^{-x}} \]  

(b) \[ \lim_{{x \to -\infty}} \frac{6e^{4x} - e^{-2x}}{8e^{4x} - e^{-2x} + 3e^{-x}} \]  

(c) \[ \lim_{{t \to -\infty}} \frac{e^{6t} - 4e^{6t}}{2e^{3t} - 5e^{-9t} + e^{-3t}} \]

**Solution**

As with the previous example, the only difference between the first two parts is that one of the limits is going to plus infinity and the other is going to minus infinity and just as with the previous example each will need to be worked differently.

(a) \[ \lim_{{x \to \infty}} \frac{6e^{4x} - e^{-2x}}{8e^{4x} - e^{2x} + 3e^{-x}} \]  

The basic concept involved in working this problem is the same as with rational expressions in the previous section. We look at the denominator and determine the exponential function with the “largest” exponent which we will then factor out from both numerator and denominator. We will use the same reasoning as we did with the previous example to determine the “largest” exponent. In the case since we are looking at a limit at plus infinity we only look at exponentials with positive exponents.

So, we’ll factor an \( e^{4x} \) out of both then numerator and denominator. Once that is done we can cancel the \( e^{4x} \) and then take the limit of the remaining terms. Here is the work for this limit,
In this case we’re going to minus infinity in the limit and so we’ll look at exponentials in the denominator with negative exponents in determining the “largest” exponent. There’s only one however in this problem so that is what we’ll use.

Again, remember to only look at the denominator. Do NOT use the exponential from the numerator, even though that one is “larger” than the exponential in the denominator. We always look only at the denominator when determining what term to factor out regardless of what is going on in the numerator.

Here is the work for this part.

\[
\lim_{x \to -\infty} \frac{6e^{4x} - e^{-2x}}{8e^{4x} - e^{2x} + 3e^{-x}} = \lim_{x \to -\infty} e^{-x} \left( \frac{6e^x - e^{-x}}{8e^x - e^{-x} + 3} \right) = \frac{6 - 0}{8 - 0 + 0} = \frac{3}{4}
\]

We’ll do the work on this part with much less detail.
Next, let’s take a quick look at some basic limits involving logarithms.

**Example 5** Evaluate each of the following limits.

\[
\lim_{x \to 0^+} \ln x \quad \text{and} \quad \lim_{x \to 0^+} \ln x
\]

**Solution**

As with the last example I’ll leave it to you to verify these restatements from the basic logarithm section.

\[
\lim_{x \to 0^+} \ln x = -\infty \quad \text{and} \quad \lim_{x \to 0^+} \ln x = \infty
\]

Note that we had to do a right-handed limit for the first one since we can’t plug negative x’s into a logarithm. This means that the normal limit won’t exist since we must look at x’s from both sides of the point in question and x’s to the left of zero are negative.

From the previous example we can see that if the argument of a log (the stuff we’re taking the log of) goes to zero from the right (i.e. always positive) then the log goes to negative infinity in the limit while if the argument goes to infinity then the log also goes to infinity in the limit.

Note as well that we can’t look at a limit of a logarithm as x approaches minus infinity since we can’t plug negative numbers into the logarithm.

Let’s take a quick look at some logarithm examples.

**Example 6** Evaluate each of the following limits.

(a) \( \lim_{x \to 0} (7x^3 - x^2 + 1) \)  
(b) \( \lim_{t \to 0} \left( \frac{1}{t^2 - 5t} \right) \)

**Solution**

(a) \( \lim_{x \to 0} (7x^3 - x^2 + 1) \)

So, let’s first look to see what the argument of the log is doing,
\[
\lim_{x \to \infty} (7x^3 - x^2 + 1) = \infty
\]

The argument of the log is going to infinity and so the log must also be going to infinity in the
limit. The answer to this part is then,
\[
\lim_{x \to \infty} \ln(7x^3 - x^2 + 1) = \infty
\]

(b) \( \lim_{t \to \infty} \ln \left( \frac{1}{t^2 - 5t} \right) \)

First, note that the limit going to negative infinity here isn’t a violation (necessarily) of the fact
that we can’t plug negative numbers into the logarithm. The real issue is whether or not the
argument of the log will be negative or not.

Using the techniques from earlier in this section we can see that,
\[
\lim_{t \to \infty} \frac{1}{t^2 - 5t} = 0
\]

and let’s also note that for negative numbers (which we can assume we’ve got since we’re going
to minus infinity in the limit) the denominator will always be positive and so the quotient will
also always be positive. Therefore, not only does the argument go to zero, it goes to zero from
the right. This is exactly what we need to do this limit.

So, the answer here is,
\[
\lim_{t \to \infty} \ln \left( \frac{1}{t^2 - 5t} \right) = -\infty
\]

As a final set of examples let’s take a look at some limits involving inverse tangents.

**Example 7** Evaluate each of the following limits.

(a) \( \lim_{x \to \infty} \tan^{-1} x \)  [Solution]

(b) \( \lim_{x \to -\infty} \tan^{-1} x \)  [Solution]

(c) \( \lim_{x \to \infty} \tan^{-1} \left( x^3 - 5x + 6 \right) \)  [Solution]

(d) \( \lim_{x \to 0} \tan^{-1} \left( \frac{1}{x} \right) \)  [Solution]

**Solution**
The first two parts here are really just the basic limits involving inverse tangents and can easily be
found by examining the following sketch of inverse tangents. The remaining two parts are more
involved but as with the exponential and logarithm limits really just refer back to the first two parts as we’ll see.

(a) \( \lim_{x \to \infty} \tan^{-1} x \)

As noted above all we really need to do here is look at the graph of the inverse tangent. Doing this shows us that we have the following value of the limit.

\[
\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}
\]

(b) \( \lim_{x \to -\infty} \tan^{-1} x \)

Again, not much to do here other than examine the graph of the inverse tangent.

\[
\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}
\]

(c) \( \lim_{x \to \infty} \tan^{-1}(x^3 - 5x + 6) \)

Okay, in part (a) above we saw that if the argument of the inverse tangent function (the stuff inside the parenthesis) goes to plus infinity then we know the value of the limit. In this case (using the techniques from the previous section) we have,

\[
\lim_{x \to \infty} x^3 - 5x + 6 = \infty
\]

So, this limit is,

\[
\lim_{x \to \infty} \tan^{-1}(x^3 - 5x + 6) = \frac{\pi}{2}
\]
(d) \( \lim_{x \to 0} \tan^{-1} \left( \frac{1}{x} \right) \)

Even though this limit is not a limit at infinity we’re still looking at the same basic idea here. We’ll use part (b) from above as a guide for this limit. We know from the Infinite Limits section that we have the following limit for the argument of this inverse tangent,

\[
\lim_{x \to 0} \frac{1}{x} = -\infty
\]

So, since the argument goes to minus infinity in the limit we know that this limit must be,

\[
\lim_{x \to 0^-} \tan^{-1} \left( \frac{1}{x} \right) = -\frac{\pi}{2}
\]

To see a precise and mathematical definition of this kind of limit see the The Definition of the Limit section at the end of this chapter.