Preface

Here are the solutions to the practice problems for my Calculus I notes. Some solutions will have more or less detail than other solutions. The level of detail in each solution will depend up on several issues. If the section is a review section, this mostly applies to problems in the first chapter, there will probably not be as much detail to the solutions given that the problems really should be review. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you’ve reached the level of working the harder problems then you will probably already understand the basics fairly well and won’t need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven’t been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.

More Optimization Problems

1. We want to construct a window whose middle is a rectangle and the top and bottom of the window are semi-circles. If we have 50 meters of framing material what are the dimensions of the window that will let in the most light?

Step 1
Let’s start with a quick sketch of the window.

1. We want to construct a window whose middle is a rectangle and the top and bottom of the window are semi-circles. If we have 50 meters of framing material what are the dimensions of the window that will let in the most light?

Step 1
Let’s start with a quick sketch of the window.
Step 2
Next we need to set up the constraint and equation that we are being asked to optimize.

We are told that we have 50 meters of framing material (i.e. the perimeter of the window) and so that will be the constraint for this problem.

\[ 50 = 2h + 2(\pi r) = 2h + 2\pi r \]

We are being asked to maximize the amount of light being let in and that is simply the enclosed area or,

\[ A = h(2r) + 2\left(\frac{1}{2}\pi r^2\right) = 2hr + \pi r^2 \]

With both of these equations we were a little careful with the last term. In each case we needed either the perimeter or area of each semicircle and there were two of them. The end result of course is the equation of the perimeter/area of a whole circle, but we really should be careful setting these equations up and note just where everything is coming from.

Step 3
Now, let’s solve the constraint for \( h \).

\[ h = 25 - \pi r \]

Plugging this into the area function gives,

\[ A(r) = 2(25 - \pi r)r + \pi r^2 = 50r - \pi r^2 \]

Step 4
Finding the critical point(s) for this shouldn’t be too difficult at this point. Here is the derivative.

\[ A'(r) = 50 - 2\pi r \]
From this it looks like we get a single critical point are: 
\[ r = \frac{25}{\pi} \approx 7.9577. \]

Step 5
The second derivative of the volume function is,
\[ A''(r) = -2\pi \]

From this we can see that the second derivative is always negative. Therefore, \( A(r) \) will always be concave down and so the single critical point we got in Step 4 must be a relative maximum and hence must be the value that allows in the maximum amount of light.

Step 6
Now, let’s finish the problem by getting the radius of the semicircles.

\[ h = 25 - \pi \left( \frac{25}{\pi} \right) = 0 \]

Okay, what this means is that in fact the most light will come from not even having a rectangle between the semicircles and just having a circular window of radius \( r = \frac{25}{\pi} \).

2. Determine the area of the largest rectangle that can be inscribed in a circle of radius 1.

Step 1
Let’s start with a quick sketch of the circle and rectangle. Also, in order to make the work a little easier we went ahead and assumed that the circle was centered at the origin of the standard \( xy \)-coordinate system.

We’ve also defined a point \((x,y)\) in the first quadrant. This is the point that we will be attempting to find when we get into the problems. If we know the coordinates of this point then the rectangle defined by the point, as shown in the figure, will be the one with the largest area.
Step 2
Next we need to set up the constraint and equation that we are being asked to optimize.

Given our graph above we can easily determine the equation of the circle. This will also be the constraint of the problem because the corners of the rectangle must be on the circle.

\[ x^2 + y^2 = 1 \]

Also note that from the figure or equation we can clearly see that \(-1 \leq x \leq 1\) and \(-1 \leq y \leq 1\). One or both of these limits will be useful later on in the problem.

We are being asked to maximize the amount of the rectangle and using the definitions we see in the figure above the area is,

\[ A = (2x)(2y) = 4xy \]

Step 3
We can solve the constraint for \(x\) or \(y\). Either will lead to essentially the same work so we’ll solve for \(x\).

\[ x = \pm \sqrt{1 - y^2} \]

Because we’ve defined the point on the circle to be in the \(1^{st}\) quadrant we will use the “+” portion of this. Plugging this into the area function gives,

\[ A(y) = 4y\sqrt{1 - y^2} \]

Step 4
Finding the critical point(s) for this shouldn’t be too difficult at this point. Here is the derivative.
\[ A'(y) = 4\sqrt{1 - y^2} - \frac{4y^2}{\sqrt{1 - y^2}} = \frac{4 - 8y^2}{\sqrt{1 - y^2}} \]

From this it looks like, from the numerator, we get the critical points,
\[ y = \pm \sqrt{\frac{1}{2}} = \pm \frac{1}{\sqrt{2}} = \pm 0.7071 \]

From the denominator we get the critical points: \( y = \pm 1 \).

Before proceeding to the next step let's notice that because our point is in the first quadrant we know that \( y \) must be positive. This fact along with the limits on \( y \) we discussed in Step 2 tells us that we must have: \( 0 \leq y \leq 1 \).

This in turn tells us that the only two critical points that we need to worry about are,
\[ y = \frac{1}{\sqrt{2}} = 0.7071 \quad \quad y = 1 \]

Step 5
Because we've got a range for possible critical points all we need to do to determine the maximum area is plug the end points and critical points into the area.
\[ A(0) = A\left(\frac{1}{\sqrt{2}}\right) = 2 \quad \quad A(1) = 0 \]

Step 6
So, the area of the largest rectangle that can be inscribed in the circle is: 2.

3. Find the point(s) on \( x = 3 - 2y^2 \) that are closest to \((-4,0)\).

Step 1
Let's start with a quick sketch of this situation. Below is a sketch of the graph of the function as well as the point \((-4,0)\). As we can see we can expect to get two points as answers with the only difference being the sign on the \( y \)-coordinate.
Step 2  
Next we need to set up the constraint and equation that we are being asked to optimize. 

In this case the constraint is simply the equation we are given. The point must lie on the graph and so must also satisfy the equation. 

\[ x = 3 - 2y^2 \]

We are being asked to minimize the distance between a point (or points) on the graph and the point \((-4, 0)\). We can do this by looking at the distance between \((-4, 0)\) and \((x, y)\). The distance between these two points is, 

\[ d = \sqrt{(x + 4)^2 + y^2} \]

As we discussed in the notes for this section the point that minimizes the square of the distance will also minimize the distance itself and so to avoid dealing with the root we will minimize the square of the distance or, 

\[ d^2 = (x + 4)^2 + y^2 \]

Step 3  
Now we have two choices on how to proceed from this point. The first option is to plug the equation we are given into the \(x\) in the distance squared and get a 4\(^{th}\) degree polynomial for \(y\) that we’ll need to work with. The second is to solve the equation for \(y^2\) and plug that into the distance squared and get a 2\(^{nd}\) degree polynomial for \(x\) that we’ll need to work with. The second option gives a “nicer” polynomial to work with so we’ll do that. 

\[ y^2 = \frac{1}{2}(3 - x) = \frac{3}{2} - \frac{1}{2}x \]
Plugging this into the distance squared gives,

\[ f(x) = d^2 = (x + 4)^2 + \frac{1}{2} - \frac{1}{2}x = x^2 + \frac{15}{2}x + \frac{35}{2} \]

Step 4
Finding the critical point(s) for this shouldn’t be too difficult at this point. Here is the derivative.

\[ f'(x) = 2x + \frac{15}{2} \]

From this it looks like we get a single critical point : \( x = -\frac{15}{4} = -3.75 \).

Step 5
The second derivative of the distance squared function is,

\[ f''(x) = 2 \]

From this we can see that the second derivative is always positive. Therefore the distance squared will always be concave up and so the single critical point we got in Step 4 must be a relative minimum and hence must be the value of \( x \) that gives the points that are closest to \((-4, 0)\).

Step 6
Finally we just need to determine the values \( y \) that give the actual points.

\[ y^2 = \frac{1}{2} - \frac{1}{2}(-\frac{15}{4}) = \frac{27}{8} \quad \Rightarrow \quad y = \pm\sqrt{\frac{27}{8}} = \pm1.8371 \]

So, the two points on the graph that are closest to \((-4, 0)\) are,

\[ (-\frac{15}{4}, \sqrt{\frac{27}{4}}) \quad \& \quad (-\frac{15}{4}, -\sqrt{\frac{27}{4}}) \]

4. An 80 cm piece of wire is cut into two pieces. One piece is bent into an equilateral triangle and the other will be bent into a rectangle with one side 4 times the length of the other side. Determine where, if anywhere, the wire should be cut to maximize the area enclosed by the two figures.

Step 1
Before we do a sketch we’ll need to do a little setup. Let’s suppose that the length of the piece of wire that goes to the rectangle is \( x \). This means that the length of the piece of wire going to the triangle is \( 80 - x \).

We know that the length of each side of the triangle are equal and so must have length \( \frac{1}{3} (80 - x) \). We also know that the interior angles of the triangle are \( \frac{\pi}{3} \) and so the height of the triangle is \( \frac{1}{3} (80 - x) \sin \left( \frac{\pi}{3} \right) = \frac{\sqrt{3}}{2} (80 - x) \).

For the rectangle let’s suppose that the length of the smaller side is \( L \) and so the length of the larger side is \( 4L \). Next, we know that the total perimeter of the rectangle is \( x \) and so we must have,

\[
x = 2L + 2(4L) = 10L \quad \rightarrow \quad L = \frac{x}{10}
\]

Now that we have all the various lengths of the figures in terms of \( x \) (which will make the work here a little easier) let’s summarize everything up with the following figure.

---

**Step 2**

Next we need to set up the constraint and equation that we are being asked to optimize.

This is one of those cases where we really don’t have a constraint equation to work with. The constraint is the length of the wire (80 cm), but we took that into account when we set up our figure above so there isn’t anything to do with that in this case.

We are being asked to maximize the enclosed area of the two figures and so here is the total area of the enclosed figures.

\[
A(x) = \left( \frac{x}{10} \right) \left( \frac{2x}{3} \right) + \frac{1}{2} \left[ \frac{1}{3} (80 - x) \right] \left[ \frac{\sqrt{3}}{2} (80 - x) \right] = \frac{x^2}{30} + \frac{\sqrt{3}}{30} (80 - x)^2
\]

**Step 3**
Finding the critical point(s) for this shouldn’t be too difficult at this point (although the Algebra will be a little messy). Here is the derivative.

$$A'(x) = \frac{2x}{25} - \frac{3x}{18} (80 - x)$$

From this it looks like we get a single critical point,

$$x = \frac{40\sqrt{2}}{9} = 43.6828$$

Step 4
The second derivative of the area function is,

$$A''(x) = \frac{2}{25} + \frac{3}{18}$$

From this we can see that the second derivative is always positive. Therefore \(A'(x)\) will always be concave up and so the single critical point we got in Step 4 must be a relative minimum and hence must be the value of \(x\) (i.e. the cut point) that will give the minimum enclosed area.

This is a problem however as we were asked for the maximum enclosed area. This is the reason for this step being in every problem that we’ve worked over the last couple of sections. Far too often students get to this point, get a single answer and then just assume that it must be the correct answer and don’t bother doing any kind of checking to verify if it is the correct answer.

After all there was a single value so there is no choice for it to be correct. Right? Well, no. As we’ll seen here it in fact is not the correct answer.

Step 5
So, what to do? We’ll recall for the problem statement that we were asked to,

“Determine where, if anywhere, the wire should be cut to maximize the area enclosed by the two figures.”

The “if anywhere” portion seems to suggest that we may not want to cut it at all. Maybe all of the wire should go to the rectangle (corresponding to \(x = 80\) above) or maybe all of the wire should to the triangle (corresponding to \(x = 0\) above).

So, all we need to do is plug \(x = 80\) and \(x = 0\) into the area function and determine which will give the largest area.
Note that we included the critical point above just to make it really clear that it will not in fact give the maximum area. We didn’t really need to include it here as we already knew it wouldn’t work for us.

From the function evaluations above it looks like we’ll need to take all of the wire and bend it into an equilateral triangle in order to get the maximum area.

5. A line through the point \((2,5)\) forms a right triangle with the \(x\)-axis and \(y\)-axis in the 1\(^{st}\) quadrant. Determine the equation of the line that will minimize the area of this triangle.

Step 1
This problem may seem a little tricky at first. Here is a sketch of a line that goes through the point \((2,5)\), has an \(x\)-intercept of \((a,0)\) and a \(y\)-intercept of \((0,b)\).

Note that the only way we can get a triangle with the line, \(x\)-axis and \(y\)-axis as sides is to require that \(a > 2\) and \(b > 5\). If either of those are not true we will not have the triangle that we want.

Step 2
Next we need to set up the constraint and equation that we are being asked to optimize.

We are being asked to minimize the area of the triangle shown above. In terms of the quantities given on the graph it is easy enough to get an equation for the area. The base length of the triangle is \(a\) and the height of the triangle is \(b\). We don’t have values for either of these but that isn’t a problem. Here is the area of the triangle.
The constraint in this case is the equation of the line since that will define the hypotenuse of the triangle and hence also give both the base and height of the triangle. We need to write down the equation of the line, but we have three points on the line that we can use. Note however, that we really should use \((2,5)\) as one of the points because the line does need to go the point and using this point to write down the equation will give us that without any extra work.

The real question then is whether we should use the \(x\) or \(y\)-intercept for the second point when determining the slope of the line. It really doesn’t matter which point that you use. The work will be slightly different for each point but there will be no real difference in the difficulty of the problem.

We are going to use \((a,0)\) for the second point. The slope of the line using this point is,

\[
m = \frac{5}{2-a}
\]

We already know that \(b\) is the \(y\)-intercept and so the equation of the line through the point is,

\[
y = \frac{5}{2-a} x + b
\]

Note that we definitely seem to have a problem here. Normally at this point we’ve got two equation and two unknowns. In this case we appear to have four unknowns : \(a, b, x\) and \(y\). This isn’t a problem as well see in the next step.

Step 3
We now need to solve the constraint for one of the unknowns in the area function, \(i.e\) either \(a\) or \(b\). However, as we noted above we also have an \(x\) and \(y\) in the equation that will cause problems if they stay in the equation.

The point of this step is to get the area function down to a single variable. If we leave the \(x\) and \(y\) in the equation of the line we will end up with an area function with not one variable but three and that won’t work for us.

What we really need is an equation involving only \(a\) and \(b\) that we can solve for one or the other and plug into the area function. Luckily this is easy to get. All we need to do is plug the \(x\)-intercept into the equation of the line to get,

\[
0 = \frac{5}{2-a} a + b
\]
Do you see why we couldn’t have used the \( y \)-intercept here? If not, plug it in and you'll very quickly see why it won’t work.

At this point we can easily solve the equation for \( b \) to get,

\[
b = -\frac{5a}{2-a} = \frac{5a}{a-2}
\]

To eliminate one of the minus signs we took the minus sign in front of the quotient and applied it to the denominator and simplified. This doesn’t need to be done, but it does eliminate one of them.

Note that if we had used the \( y \)-intercept to determine the slope we would have found it to be easier at this step to solve for \( a \) instead. That is the only real difference in which point you use to find the slope.

Okay, let’s put all this together. We know the value of \( b \) in terms of \( a \) so plug that into the area function to get,

\[
A(a) = \frac{1}{2}(a)\left(\frac{5a}{a-2}\right) = \frac{5}{2} \frac{a^2}{a-2}
\]

Step 4
Here is the derivative of the area function,

\[
A'(a) = \frac{5}{2} \frac{a^2 - 4a}{(a-2)^2} = \frac{5}{2} \frac{a(a-4)}{(a-2)^2}
\]

From this it looks like we get a three possible critical points: \( a = 0 \), \( a = 2 \) and \( a = 4 \).

We can’t use \( a = 0 \) as the critical point because that will no longer form a triangle with both the \( x \)-axis and the \( y \)-axis as the problem asks for as noted in the first step.

We also can’t use \( a = 2 \) for two reasons. First, it isn’t actually a critical point because the area function doesn’t exist at \( a = 2 \). This shouldn’t be surprising given that if we used this point we wouldn’t have a triangle anyway (again as we noted in the first step) and that is also the second reason for not using it.

This leaves only \( a = 4 \) as a potential critical point that we can use.

Step 5
The second derivative of the area function (after a little simplification) is,
From this we can see that the second derivative is always positive provided we have \( a > 2 \). However, as we noted in the first step this is required in order even work the problem. Therefore, the second derivative will always be positive for the range of \( a \) that we are working on. The area function will then will always be concave up for the range of \( a \) and \( a = 4 \) must give a minimum area.

Step 6
Now that we know the value of \( a \) we know that the slope and \( y \)-intercept are,

\[
m = \frac{5}{2 - 4} = -\frac{5}{2} \quad \quad b = \frac{5(4)}{4 - 2} = 10
\]

The equation of the line is then,

\[
y = -\frac{5}{2}x + 10
\]

6. A piece of pipe is being carried down a hallway that is 18 feet wide. At the end of the hallway there is a right-angled turn and the hallway narrows down to 12 feet wide. What is the longest pipe (always keeping it horizontal) that can be carried around the turn in the hallway?

Step 1
Let’s start with a quick sketch of the pipe and hallways with all the important quantities given.

Step 2
Next we need to set up the constraint and equation that we are being asked to optimize.
As we discussed in the similar problem in the notes for this section we actually need to minimize the total length of the pipe. The equation we need to minimize is then,

\[ L = L_1 + L_2 \]

Also as we discussed in the notes problem with actually have two constraints: the widths of the two hallways. We can easily solve for these in terms of the angle \( \theta \).

\[ L_1 = 12 \sec \theta \quad L_2 = 18 \csc \theta \]

As discussed in the notes problem we also know that we must have \( 0 < \theta < \frac{\pi}{2} \).

Step 3
All we need to do here is plug our two constraints in the length function to get a function in terms of \( \theta \) that we can minimize.

\[ L(\theta) = 12 \sec \theta + 18 \csc \theta \]

Step 4
The derivative of the length function is,

\[ L'(\theta) = 12 \sec \theta \tan \theta - 18 \csc \theta \cot \theta \]

Next we need to set this equal to zero and solve this for \( \theta \) to get the critical point that is in the range \( 0 < \theta < \frac{\pi}{2} \).

\[ 12 \sec \theta \tan \theta = 18 \csc \theta \cot \theta \]

\[ \sec \theta \tan \theta = \frac{18}{12} = \frac{3}{2} \]

\[ \csc \theta \cot \theta = 3 \]

\[ \tan^3 \theta = \frac{3}{2} \]

The critical point that we need is then: \( \theta = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right) = 0.8528 \).

Step 5
Verifying that this is the value that gives the minimum is a little trickier than the other problems.

As noted in the notes for this section as we move \( \theta \to 0 \) we have \( L \to \infty \) and as we move \( \theta \to \frac{\pi}{2} \) we have \( L \to \infty \). Therefore, on either side of \( \theta = 0.8528 \) radians the length of the pipe is increasing to infinity as we move towards the end of the range.
Therefore, this angle must give us the minimum length of the pipe and so is the largest pipe that we can fit around corner.

Step 6
The largest pipe that we can fit around the corner is then,

\[ L(0.8528) = 42.1409 \text{ feet} \]

7. Two 10 meter tall poles are 30 meters apart. A length of wire is attached to the top of each pole and it is staked to the ground somewhere between the two poles. Where should the wire be staked so that the minimum amount of wire is used?

Step 1
Let’s start with a quick of the situation.

Step 2
Next we need to set up the constraint and equation that we are being asked to optimize.

We want to minimize the amount of wire and so the equation we need to minimize is,

\[ L = L_1 + L_2 \]

The constraint here is that the poles must be 30 meters apart. We can use this to determine the lengths of the individual wires in terms of \( x \). Doing this gives,

\[ L_1 = \sqrt{100 + x^2} \quad L_2 = \sqrt{100 + (30 - x)^2} \]
Note as well that can also see that we need to require that $0 \leq x \leq 30$.

Step 3
All we need to do here is plug the lengths of the individual wires in the total length to get a function in terms of $x$ that we can minimize.

$$L(x) = \sqrt{100 + x^2} + \sqrt{100 + (30 - x)^2}$$

Step 4
The derivative of the length function is,

$$L'(x) = \frac{x}{\sqrt{100 + x^2}} + \frac{x - 30}{\sqrt{x^2 - 60x + 1000}}$$

Solving for the critical point(s) is going to be messy so here it goes.

$$\frac{x}{\sqrt{100 + x^2}} + \frac{x - 30}{\sqrt{x^2 - 60x + 1000}} = 0$$

$$\frac{x}{\sqrt{100 + x^2}} = -\frac{x - 30}{\sqrt{x^2 - 60x + 1000}}$$

$$x \sqrt{x^2 - 60x + 1000} = -(x - 30) \sqrt{100 + x^2}$$

$$x^2 (x^2 - 60x + 1000) = (x - 30)^2 (100 + x^2)$$

$$x^4 - 60x^3 + 1000x^2 = x^4 - 60x^3 + 1000x^2 - 6000x + 90000$$

$$0 = -6000x + 90000$$

$$x = 15$$

A quick check by plugging this back into the derivative shows that we do indeed get $L'(15) = 0$ and so this is a critical point and it is in the acceptable range of $x$.

Recall that because we squared both sides of the equation above it is possible to end up with answers that in fact are not solutions and so we have to go back and check in the original equation to make sure that they are solutions.

Step 5
Since we have a range of $x$'s and the distance function is continuous in the range all we need to do is plug in the endpoints and the critical point to identify the minimum distance.

$$L(0) = 41.6228 \quad L(15) = 36.0555 \quad L(30) = 41.6228$$
Step 6
The wire should be staked midway between the poles to minimize the amount of wire.