Here are my online notes for my Calculus I course that I teach here at Lamar University. Despite the fact that these are my “class notes” they should be accessible to anyone wanting to learn Calculus I or needing a refresher in some of the early topics in calculus.

I’ve tried to make these notes as self contained as possible and so all the information needed to read through them is either from an Algebra or Trig class or contained in other sections of the notes.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. Because I want these notes to provide some more examples for you to read through, I don’t always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than are presented here.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible when writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
**Proof of Trig Limits**

In this section we’re going to provide the proof of the two limits that are used in the derivation of the derivative of sine and cosine in the Derivatives of Trig Functions section of the Derivatives chapter.

**Proof of:** \[ \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \]

This proof of this limit uses the Squeeze Theorem. However, getting things set up to use the Squeeze Theorem can be a somewhat complex geometric argument that can be difficult to follow so we’ll try to take it fairly slow.

Let’s start by assuming that \( 0 \leq \theta \leq \frac{\pi}{2} \). Since we are proving a limit that has \( \theta \to 0 \) it’s okay to assume that \( \theta \) is not too large (i.e. \( \theta \leq \frac{\pi}{2} \)). Also, by assuming that \( \theta \) is positive we’re actually going to first prove that the above limit is true if it is the right-hand limit. As you’ll see if we can prove this then the proof of the limit will be easy.

So, now that we’ve got our assumption on \( \theta \) taken care of let’s start off with the unit circle circumscribed by an octagon with a small slice marked out as shown below.

Points \( A \) and \( C \) are the midpoints of their respective sides on the octagon and are in fact tangent to the circle at that point. We’ll call the point where these two sides meet \( B \).

From this figure we can see that the circumference of the circle is less than the length of the octagon. This also means that if we look at the slice of the figure marked out above then the length of the portion of the circle included in the slice must be less than the length of the portion of the octagon included in the slice.

Because we’re going to be doing most of our work on just the slice of the figure let’s strip that out and look at just it. Here is a sketch of just the slice.
Now denote the portion of the circle by arc $AC$ and the lengths of the two portion of the octagon shown by $|AB|$ and $|BC|$. Then by the observation about lengths we made above we must have,

$$\text{arc } AC < |AB| + |BC|$$

(1)

Next, extend the lines $AB$ and $OC$ as shown below and call the point that they meet $D$. The triangle now formed by $AOD$ is a right triangle. All this is shown in the figure below.

The triangle $BCD$ is a right triangle with hypotenuse $BD$ and so we know $|BC| < |BD|$. Also notice that $|AB| + |BD| = |AD|$. If we use these two facts in (1) we get,

$$\text{arc } AC < |AB| + |BC|$$

$$< |AB| + |BD|$$

$$= |AD|$$

(2)

Next, as noted already the triangle $AOD$ is a right triangle and so we can use a little right triangle trigonometry to write $|AD| = |AO| \tan \theta$. Also note that $|AO| = 1$ since it is nothing more than the radius of the unit circle. Using this information in (2) gives,

$$\text{arc } AC < |AD|$$

$$< |AO| \tan \theta$$

$$= \tan \theta$$

(3)
The next thing that we need to recall is that the length of a portion of a circle is given by the radius of the circle times the angle that traces out the portion of the circle we’re trying to measure. For our portion this means that,

$$\text{arc } AC = |AO| \theta = \theta$$

So, putting this into (3) we see that,

$$\theta = \text{arc } AC < \tan \theta = \frac{\sin \theta}{\cos \theta}$$

or, if we do a little rearranging we get,

$$\cos \theta < \frac{\sin \theta}{\theta} \tag{4}$$

We’ll be coming back to (4) in a bit. Let’s now add in a couple more lines into our figure above. Let’s connect $A$ and $C$ with a line and drop a line straight down from $C$ until it intersects $AO$ at a right angle and let’s call the intersection point $E$. This is all shown in the figure below.

Okay, the first thing to notice here is that,

$$|CE| < |AC| < \text{arc } AC \tag{5}$$

Also note that triangle $EOC$ is a right triangle with a hypotenuse of $|CO| = 1$. Using some right triangle trig we can see that,

$$|CE| = |CO| \sin \theta = \sin \theta$$

Plugging this into (5) and recalling that $\text{arc } AC = \theta$ we get,

$$\sin \theta = |CE| < \text{arc } AC = \theta$$

and with a little rewriting we get,

$$\frac{\sin \theta}{\theta} < 1 \tag{6}$$
Okay, we’re almost done here. Putting (4) and (6) together we see that,
\[ \cos \theta < \frac{\sin \theta}{\theta} < 1 \]
provided \( 0 \leq \theta \leq \frac{\pi}{2} \). Let’s also note that,
\[ \lim_{\theta \to 0} \cos \theta = 1 \quad \text{and} \quad \lim_{\theta \to 0} 1 = 1 \]

We are now set up to use the Squeeze Theorem. The only issue that we need to worry about is that we are staying to the right of \( \theta = 0 \) in our assumptions and so the best that the Squeeze Theorem will tell us is,
\[ \lim_{\theta \to 0^+} \frac{\sin \theta}{\theta} = 1 \]

So, we know that the limit is true if we are only working with a right-hand limit. However we know that \( \sin \theta \) is an odd function and so,
\[ \frac{\sin(-\theta)}{-\theta} = -\frac{\sin \theta}{-\theta} = \frac{\sin \theta}{\theta} \]

In other words, if we approach zero from the left (\( i.e. \) negative \( \theta \)’s) then we’ll get the same values in the function as if we’d approached zero from the right (\( i.e. \) positive \( \theta \)’s) and so,
\[ \lim_{\theta \to 0^-} \frac{\sin \theta}{\theta} = 1 \]

We have now shown that the two one-sided limits are the same and so we must also have,
\[ \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \]

That was a somewhat long proof and if you’re not really good at geometric arguments it can be kind of daunting and confusing. Nicely, the second limit is very simple to prove, provided you’ve already proved the first limit.

\[ \lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0 \]

Proof of:
\[ \lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0 \]

We’ll start by doing the following,
\[ \lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = \lim_{\theta \to 0} \frac{(\cos \theta - 1)(\cos \theta + 1)}{\theta(\cos \theta + 1)} = \lim_{\theta \to 0} \frac{\cos^2 \theta - 1}{\theta(\cos \theta + 1)} \quad (7) \]

Now, let’s recall that,
\[
cos^2 \theta + \sin^2 \theta = 1 \quad \Rightarrow \quad \cos^2 \theta - 1 = -\sin^2 \theta
\]

Using this in (7) gives us,

\[
\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = \lim_{\theta \to 0} \frac{-\sin^2 \theta}{\theta (\cos \theta + 1)}
= \lim_{\theta \to 0} \frac{\sin \theta - \sin \theta}{\theta (\cos \theta + 1)}
= \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \lim_{\theta \to 0} \frac{-\sin \theta}{\cos \theta + 1}
\]

At this point, because we just proved the first limit and the second can be taken directly we’re pretty much done. All we need to do is take the limits.

\[
\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \lim_{\theta \to 0} \frac{-\sin \theta}{\cos \theta + 1} = (1)(0) = 0
\]