Preface

Here are the solutions to the practice problems for my Calculus I notes. Some solutions will have more or less detail than other solutions. The level of detail in each solution will depend up on several issues. If the section is a review section, this mostly applies to problems in the first chapter, there will probably not be as much detail to the solutions given that the problems really should be review. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you’ve reached the level of working the harder problems then you will probably already understand the basics fairly well and won’t need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven’t been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.

More Volume Problems

1. Find the volume of a pyramid of height $h$ whose base is an equilateral triangle of length $L$.

Hint : If possible, try to get a sketch of what the pyramid looks like. These can be difficult to sketch on occasion but if we can get a sketch it will help to set up the problem.

Step 1
Okay, let’s start with a sketch of the pyramid. These can be difficult to sketch, but having the sketch will help greatly with the set up portion of the problem.
We’ve got several sketches here. In each sketch we’ve shown a representative cross-sectional area (shown in red). Because the cross-section can be placed at any point on the $y$-axis the area of the cross-section will be a function of $y$ as indicated in the image.

The sketch in the upper right we see the pyramid from the “front” and the sketch in the upper left we see pyramid from the “top”. Note that we set the point of the pyramid at the origin and drew the pyramid upwards. This was done to make the set up for the problem a little easier. Also we sketched the pyramid so that one of the sides of the pyramid was parallel to the $x$-axis. This was done only so we could draw in the bottom sketch (which we’ll get to in a second) and have the images match up, so to speak.

The bottom sketch is a sketch of the side of the pyramid that is parallel to the $x$-axis. It also has all of the various quantities that we’ll need shown. The representative cross-section here is indicated by the red line on the sketch.
Hint: Determine a formula for the cross-sectional area in terms of $y$.

Step 2
Let’s start off with a sketch of what a typical cross-section looks like.

In this case we know that the cross-sections are equilateral triangles and so all of the interior angles are $\frac{\pi}{3}$ and we know that all the sides are the same length, let’s say $s$. In the sketch above notice that since we have an equilateral triangle we know that the dashed line (representing the height of the triangle) will divide the base of the triangle into equal length portions, i.e. $\frac{s}{2}$. Also from basic right triangle trig (each “half” of the cross-section is a right triangle right?) we can see that we can write the height in terms of $s$ as follows,

$$\tan\left(\frac{\pi}{3}\right) = \frac{h}{\frac{s}{2}} \Rightarrow h = \frac{s}{2} \tan\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} s$$

Therefore, in terms of $s$ the area of each cross-section is,

$$\text{Area} = \frac{1}{2}(s)\left(\frac{\sqrt{3}}{2} s\right) = \frac{\sqrt{3}}{4} s^2$$

Now, we know from the sketches in Step 1 that the cross-sectional area should be a function of $y$. So, if we could determine a relationship between $s$ and $y$ we’d have what we need. Let’s revisit one of the sketches from Step 1.
From this we can see that we have two similar triangles. The overall side (base $L$ and height $h$) as well as the “lower” portion formed by the red line representing the cross-sectional area (base $s$ and height $y$).

Because these two triangles are similar triangles we know the following ratios must be equal.

$$\frac{s}{y} = \frac{L}{h} \quad \Rightarrow \quad s = \frac{L}{h} y$$

We now have a relationship between $s$ and $y$ so plug this into the area formula from above to get the area of the cross-section in terms of $y$.

$$A(y) = \frac{\sqrt{3}}{4} \left( \frac{L}{h} y \right)^2 = \frac{\sqrt{3}L^2}{4h^2} y^2$$

Hint : All we need to do now is determine the volume itself.

Step 3
Finally we need the volume itself. We know that the volume is found by evaluating the following integral.

$$V = \int_{c}^{d} A(y) dy$$

We already have a formula for $A(y)$ from Step 2 and from the sketches in Step 1 we can see that the “first” cross-section will occur at $y = 0$ and that the “last” cross-section will occur at $y = h$ and so these are the limits for the integral.
The volume is then,

\[ V = \int_0^h \frac{\sqrt{3}L^2}{4h^2} y^2 \, dy = \frac{\sqrt{3}L^2}{4h^2} \int_0^h y^2 \, dy = \frac{\sqrt{3}L^2}{4h^2} \left( \frac{y^3}{3} \right) \bigg|_0^h = \frac{\sqrt{3}L^2 h}{12} \]

Do not get excited about the \( h \) and \( L \) in the integral and area formula. These are just constants. The only letter that is actually changing is \( y \). Because the \( h \) and \( L \) are constants we can factor them out of the integral as we did with the actual numbers.

2. Find the volume of the solid whose base is a disk of radius \( r \) and whose cross-sections are squares. See figure below to see a sketch of the cross-sections.

Hint : While it’s not strictly needed for this problem a sketch of the solid might be interesting to see just what the solid looks like.

Step 1
Here are a couple of sketches of the solid from three different angles. For reference the positive \( x \)-axis and positive \( y \)-axis are shown.
Because the cross-section is perpendicular to the $y$-axis as we move the cross-section along the $y$-axis we’ll change its area and so the cross-sectional area will be a function of $y$, i.e. $A(y)$.

**Hint:** Determine a formula for the cross-sectional area in terms of $y$.

**Step 2**
While the sketches above are nice to get a feel for what the solid looks like, what we really need is just a sketch of the cross-section. So, here’s a couple of sketches of the cross-sectional area.

\[ x^2 + y^2 = r^2 \]

\[ x = -\sqrt{r^2 - y^2}, \quad x = \sqrt{r^2 - y^2} \]
The sketch on the left is really just the graph given in the problem statement with the only difference that we colored the right/left sides so it will match with the sketch on the right. The sketch on the right looks at the cross-section from directly above and is shown by the red line.

Let’s get a quick sketch of just the cross-section and let’s call the length of the side of each square $s$.

Now, along the bottom we’ve denoted the $y$-axis location in the cross-section with a black dot and the orange and green dots represent where the left and right portions of the circle are at. We can also see that, assuming the cross-section is placed at some $y$, the green dot must be a distance of $\sqrt{r^2 - y^2}$ from the $y$-axis. Likewise, the orange dot must also be a distance of $\sqrt{r^2 - y^2}$ from the $y$-axis (recall we want the distance to be positive here and so we drop the minus sign from the function to get a positive distance).

Now, we know that the area of the square is simply $s^2$ and from the discussion above we see that,

\[ \frac{s}{2} = \sqrt{r^2 - y^2} \quad \Rightarrow \quad s = 2\sqrt{r^2 - y^2} \]

So, a formula for the area of the cross-section in terms of $y$ is,

\[ A(y) = s^2 = \left(2\sqrt{r^2 - y^2}\right)^2 = 4\left(r^2 - y^2\right) \]

Hint: All we need to do now is determine the volume itself.

Step 3
Finally we need the volume itself. We know that the volume is found by evaluating the following integral.

\[ V = \int_e^d A(y) \, dy \]
We already have a formula for $A(y)$ from Step 2 and from the sketches in Step 1 we can see that the “first” cross-section will occur at $y = -r$ and that the “last” cross-section will occur at $y = r$ and so these are the limits for the integral.

The volume is then,

$$V = \int_{-r}^{r} 4 \left(r^2 - y^2\right) dy = 4 \left[yr^2 - \frac{1}{3} y^3\right]_{-r}^{r} = \frac{16}{3} r^3$$

Do not get excited about the $r$ integral and area formula. It is just a constant. The only letter that is actually changing is $y$.

3. Find the volume of the solid whose base is the region bounded by $x = 2 - y^2$ and $x = y^2 - 2$ and whose cross-sections are isosceles triangles with the base perpendicular to the $y$-axis and the angle between the base and the two sides of equal length is $\frac{\pi}{4}$. See figure below to see a sketch of the cross-sections.

Hint: While it’s not strictly needed for this problem a sketch of the solid might be interesting to see just what the solid looks like.

Step 1
Here are a couple of sketches of the solid from three different angles. For reference the positive $x$-axis and positive $y$-axis are shown.
Because the cross-section is perpendicular to the $y$-axis as we move the cross-section along the $y$-axis we’ll change its area and so the cross-sectional area will be a function of $y$, i.e. $A(y)$.

Hint: Determine a formula for the cross-sectional area in terms of $y$.

Step 2
While the sketches above are nice to get a feel for what the solid looks like, what we really need is just a sketch of the cross-section. So, here’s a couple of sketches of the cross-sectional area.
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The sketch on the top is really just the graph given in the problem statement that is included for a reference with the sketch on the bottom. The sketch on the bottom looks at the cross-section from directly above and is shown by the red line.

Let's get a quick sketch of just the cross-section and let's call the length of the base of triangle $b$ and the height of the triangle $h$.

Now, along the bottom we’ve denoted the $y$-axis location in the cross-section with a black dot and the orange and green dots represent the left and right curves that define the left and right sides of the bottom of the solid. We can also see that, assuming the cross-section is placed at some $y$, the
green dot must be a distance of \( 2 - y^2 \) from the \( y \)-axis. Likewise, the orange dot must also be a distance of \(- (y^2 - 2) = 2 - y^2\) from the \( y \)-axis (recall we want the distance to be positive here and so we add the minus sign to the function to get a positive distance).

Now, we can see that the base of the triangle is given by,
\[
\frac{b}{y} = 2 - y^2 \quad \Rightarrow \quad b = 2(2 - y^2)
\]

Likewise, the height can be found from basic right triangle trig.
\[
\tan \left( \frac{\pi}{4} \right) = \frac{h}{\sqrt{2}} \quad \Rightarrow \quad h = \frac{b}{2} \tan \left( \frac{\pi}{4} \right) = 2 - y^2
\]

So, a formula for the area of the cross-section in terms of \( y \) is then,
\[
A(y) = \frac{1}{2} bh = \left( 2 - y^2 \right)^2 = 4 - 4y^2 + y^4
\]

Hint: All we need to do now is determine the volume itself.

Step 3
Finally we need the volume itself. We know that the volume is found by evaluating the following integral.
\[
V = \int_{c}^{d} A(y) \, dy
\]

By setting \( x = 0 \) into either of the equations defining the left and right sides of the base of the solid (since they intersect at the \( y \)-axis) we can see that the “first” cross-section will occur at \( y = -\sqrt{2} \) and the “last” cross-section will occur at \( y = \sqrt{2} \) and so these are the limits for the integral.

The volume is then,
\[
V = \int_{-\sqrt{2}}^{\sqrt{2}} 4 - 4y^2 + y^4 \, dy = \left[ 4y - \frac{4}{3} y^3 + \frac{1}{5} y^5 \right]_{-\sqrt{2}}^{\sqrt{2}} = \frac{8\sqrt{2}}{15}
\]

4. Find the volume of a wedge cut out of a “cylinder” whose base is the region bounded by \( y = \sqrt{4 - x} \), \( x = -4 \) and the \( x \)-axis. The angle between the top and bottom of the wedge is \( \frac{\pi}{2} \).

See the figure below for a sketch of the “cylinder” and the wedge (the positive \( x \)-axis and positive \( y \)-axis are shown in the sketch – they are just in a different orientation).
Step 1
While not strictly needed let’s redo the sketch of the “cylinder” and wedge from the problem statement only this time let’s also sketch in what the cross-section will look like.
Because the cross-section is perpendicular to the $x$-axis as we move the cross-section along the $x$-axis we’ll change its area and so the cross-sectional area will be a function of $x$, i.e. $A(x)$. Also note that as shown in the sketches the cross-section will be a right triangle.

Hint: Determine a formula for the cross-sectional area in terms of $x$.

Step 2
While the sketches above are nice to get a feel for what the solid and cross-sections look like, what we really need is just a sketch of just the cross-section. So, here are a couple of sketches of the cross-sectional area.

The sketch on the left is just pretty much the sketch we’ve seen before and is included to give us a reference point for the actual cross-section that is shown on the right.

As noted in the sketch on the right we’ll call the base of the triangle $b$ and the height of the triangle $h$. Also, the dot on the left side of the base represents where the $x$-axis is on the cross-section and the dot on the right side of the base represents the curve that defines the edge of the solid (and hence the wedge).

From this sketch it should then be pretty clear that the length of the base is simply the distance from the $x$-axis to the curve or,

$$b = \sqrt{4 - x}$$

Likewise, the height can be found from basic right triangle trig.

$$\tan \left( \frac{\pi}{3} \right) = \frac{h}{b} \quad \Rightarrow \quad h = b \tan \left( \frac{\pi}{3} \right) = \sqrt{3\sqrt{4 - x}}$$

So, a formula for the area of the cross-section in terms of $x$ is then,
\[ A(y) = \frac{1}{2} bh = \frac{\sqrt{5}}{2} \left( \sqrt{4 - x} \right)^2 = \frac{\sqrt{5}}{2} (4 - x) \]

Hint: All we need to do now is determine the volume itself.

Step 3
Finally we need the volume itself. We know that the volume is found by evaluating the following integral.

\[ V = \int_{a}^{b} A(x) \, dx \]

From the sketches in the problem statement or from Step 1 we can see that the “first” cross-section will occur at \( x = -4 \) (the back end of the “cylinder”) and the “last” cross-section will occur at \( x = 4 \) (the front end of the “cylinder” where the curve intersects with the \( x \)-axis). These are then the limits for the integral.

The volume is then,

\[ V = \int_{-4}^{4} \frac{\sqrt{5}}{2} (4 - x) \, dx = \frac{\sqrt{5}}{2} \left( 4x - \frac{1}{2} x^2 \right) \bigg|_{-4}^{4} = 16\sqrt{3} \]