Preface

Here are my online notes for my differential equations course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn how to solve differential equations or needing a refresher on differential equations.

I’ve tried to make these notes as self contained as possible and so all the information needed to read through them is either from a Calculus or Algebra class or contained in other sections of the notes.

A couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn differential equations I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Differential Equation many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don’t have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren’t worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
**Basic Concepts for n\textsuperscript{th} Order Linear Equations**

We’ll start this chapter off with the material that most text books will cover in this chapter. We will take the material from the **Second Order** chapter and expand it out to \( n \)\textsuperscript{th} order linear differential equations. As we’ll see almost all of the 2\textsuperscript{nd} order material will very naturally extend out to \( n \)\textsuperscript{th} order with only a little bit of new material.

So, let’s start things off here with some basic concepts for \( n \)\textsuperscript{th} order linear differential equations. The most general \( n \)\textsuperscript{th} order linear differential equation is,

\[
P_n(t) y^{(n)} + P_{n-1}(t) y^{(n-1)} + \cdots + P_1(t) y' + P_0(t) y = G(t)
\]

where you’ll hopefully recall that,

\[
y^{(m)} = \frac{d^m y}{dx^m}
\]

Many of the theorems and ideas for this material require that \( y^{(n)} \) has a coefficient of 1 and so if we divide out by \( P_n(t) \) we get,

\[
y^{(n)} + p_{n-1}(t) y^{(n-1)} + \cdots + p_1(t) y' + p_0(t) y = g(t)
\]

As we might suspect an IVP for an \( n \)\textsuperscript{th} order differential equation will require the following \( n \) initial conditions.

\[
y(t_0) = \bar{y}_0, \quad y'(t_0) = \bar{y}_1, \quad \cdots, \quad y^{(n-1)}(t_0) = \bar{y}_{n-1}
\]

The following theorem tells us when we can expect there to be a unique solution to the IVP given by (2) and (3).

**Theorem 1**

Suppose the functions \( p_0, p_1, \ldots, p_{n-1} \) and \( g(t) \) are all continuous in some open interval \( I \) containing \( t_0 \) then there is a unique solution to the IVP given by (2) and (3) and the solution will exist for all \( t \) in \( I \).

This theorem is a very natural extension of a similar theorem we saw in the 1\textsuperscript{st} order material.

Next we need to move into a discussion of the \( n \)\textsuperscript{th} order linear homogeneous differential equation,

\[
y^{(n)} + p_{n-1}(t) y^{(n-1)} + \cdots + p_1(t) y' + p_0(t) y = 0
\]

Let’s suppose that we know \( y_1(t), y_2(t), \ldots, y_n(t) \) are all solutions to (4) then by the an extension of the **Principle of Superposition** we know that

\[
y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)
\]

will also be a solution to (4). The real question here is whether or not this will form a general solution to (4).
In order for this to be a general solution then we will have to be able to find constants \( c_1, c_2, \ldots, c_n \) for any choice of \( t_0 \) (in the interval \( I \) from Theorem 1) and any choice of \( y_1, y_2, \ldots, y_n \). Or, in other words we need to be able to find \( c_1, c_2, \ldots, c_n \) that will solve,

\[
\begin{align*}
&c_1 y_1(t_0) + c_2 y_2(t_0) + \cdots + c_n y_n(t_0) = y_0 \\
&c_1 y_1'(t_0) + c_2 y_2'(t_0) + \cdots + c_n y_n'(t_0) = y_1 \\
&\vdots \\
&c_1 y_1^{(n-1)}(t_0) + c_2 y_2^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) = y_{n-1}
\end{align*}
\]

Just as we did for 2\(^{nd}\) order differential equations, we can use Cramer’s Rule to solve this and the denominator of each the answers will be the following determinant of an \( n \times n \) matrix.

\[
\begin{vmatrix}
  y_1 & y_2 & \cdots & y_n \\
  y_1' & y_2' & \cdots & y_n' \\
  \vdots & \vdots & \ddots & \vdots \\
  y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)}
\end{vmatrix}
\]

As we did back with the 2\(^{nd}\) order material we’ll define this to be the Wronskian and denote it by,

\[
W(y_1, y_2, \ldots, y_n)(t) = \begin{vmatrix}
  y_1 & y_2 & \cdots & y_n \\
  y_1' & y_2' & \cdots & y_n' \\
  \vdots & \vdots & \ddots & \vdots \\
  y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)}
\end{vmatrix}
\]

Now that we have the definition of the Wronskian out of the way we need to get back to the question at hand. Because the Wronskian is the denominator in the solution to each of the \( c_i \) we can see that we’ll have a solution provided it is not zero for any value of \( t = t_0 \) that we chose to evaluate the Wronskian at. The following theorem summarizes all this up.

**Theorem 2**

Suppose the functions \( p_0, p_1, \ldots, p_{n-1} \) are all continuous on the open interval \( I \) and further suppose that \( y_1(t), y_2(t), \ldots, y_n(t) \) are all solutions to (4). If \( W(y_1, y_2, \ldots, y_n)(t) \neq 0 \) for every \( t \) in \( I \) then \( y_1(t), y_2(t), \ldots, y_n(t) \) form a **Fundamental Set of Solutions** and the general solution to (4) is,

\[
y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)
\]

Recall as well that if a set of solutions form a fundamental set of solutions then they will also be a set of **linearly independent functions**.

We’ll close this section off with a quick reminder of how we find solutions to the nonhomogeneous differential equation, (2). We first need the \( n^{th} \) order version of a theorem we saw back in the 2\(^{nd}\) order material.
Theorem 3

Suppose that \(Y_1(t)\) and \(Y_2(t)\) are two solutions to (2) and that \(y_1(t), y_2(t), \ldots, y_n(t)\) are a fundamental set of solutions to the homogeneous differential equation (4) then,

\[ Y_1(t) - Y_2(t) \]

is a solution to (4) and it can be written as

\[ Y_1(t) - Y_2(t) = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t) \]

Now, just as we did with the 2\textsuperscript{nd} order material if we let \(Y(t)\) be the general solution to (2) and if we let \(Y_p(t)\) be any solution to (2) then using the result of this theorem we see that we must have,

\[ Y(t) = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t) + Y_p(t) = y_c(t) + Y_p(t) \]

where, \(y_c(t) = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t)\) is called the \textbf{complementary solution} and \(Y_p(t)\) is called a \textbf{particular solution}.

Over the course of the next couple of sections we’ll discuss the differences in finding the complementary and particular solutions for \(n\textsuperscript{th}\) order differential equations in relation to what we know about 2\textsuperscript{nd} order differential equations. We’ll see that, for the most part, the methods are the same. The amount of work involved however will often be significantly more.