Here are my online notes for my differential equations course that I teach here at Lamar University. Despite the fact that these are my “class notes” they should be accessible to anyone wanting to learn how to solve differential equations or needing a refresher on differential equations.

I’ve tried to make these notes as self contained as possible and so all the information needed to read through them is either from a Calculus or Algebra class or contained in other sections of the notes.

A couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn differential equations I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Differential Equation many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don’t have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren’t worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
Vibrating String

This will be the final partial differential equation that we’ll be solving in this chapter. In this section we’ll be solving the 1-D wave equation to determine the displacement of a vibrating string. There really isn’t much in the way of introduction to do here so let’s just jump straight into the example.

Example 1  Find a solution to the following partial differential equation.

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}
\]

\[
u(x,0) = f(x) \quad \quad \frac{\partial u}{\partial t}(x,0) = g(x)
\]

\[
u(0,t) = 0 \quad \quad u(L,t) = 0
\]

Solution

One of the main differences here that we’re going to have to deal with is the fact that we’ve now got two initial conditions. That is not something we’ve seen to this point, but will not be all that difficult to deal with when the time rolls around.

We’ve already done the separation of variables for this problem, but let’s go ahead and redo it here so we can say we’ve got another problem almost completely worked out.

So, let’s start off with the product solution.

\[
u(x,t) = \varphi(x)h(t)
\]

Plugging this into the two boundary conditions gives,

\[
\varphi(0) = 0 \quad \quad \varphi(L) = 0
\]

Plugging the product solution into the differential equation, separating and introducing a separation constant gives,

\[
\frac{\partial^2}{\partial t^2} (\varphi(x)h(t)) = c^2 \frac{\partial^2}{\partial x^2} (\varphi(x)h(t))
\]

\[
\varphi(x) \frac{d^2 h}{dt^2} = c^2 h(t) \frac{d^2 \varphi}{dx^2}
\]

\[
\frac{1}{c^2 h} \frac{d^2 h}{dt^2} = \frac{1}{\varphi} \frac{d^2 \varphi}{dx^2} = -\lambda
\]

We moved the \(c^2\) to the left side for convenience and chose \(-\lambda\) for the separation constant so the differential equation for \(\varphi\) would match a known (and solved) case.

The two ordinary differential equations we get from separation of variables are then,

\[
\frac{d^2 h}{dt^2} + c^2 \lambda h = 0 \quad \quad \frac{d^2 \varphi}{dx^2} + \lambda \varphi
\]

\[
\varphi(0) = 0 \quad \varphi(L) = 0
\]
We solved the boundary value problem above in Example 1 of the Solving the Heat Equation section of this chapter and so the eigenvalues and eigenfunctions for this problem are,

\[ \lambda_n = \left( \frac{n\pi}{L} \right)^2 \quad \varphi_n(x) = \sin \left( \frac{n\pi x}{L} \right) \quad n = 1, 2, 3, \ldots \]

The first ordinary differential equation is now,

\[ \frac{d^2 h}{dt^2} + \left( \frac{n\pi c}{L} \right)^2 h = 0 \]

and because the coefficient of the \( h \) is clearly positive the solution to this is,

\[ h(t) = c_1 \cos \left( \frac{n\pi ct}{L} \right) + c_2 \sin \left( \frac{n\pi ct}{L} \right) \]

Because there is no reason to think that either of the coefficients above are zero we then get two product solutions,

\[ u_n(x, t) = A_n \cos \left( \frac{n\pi ct}{L} \right) \sin \left( \frac{n\pi x}{L} \right) \quad n = 1, 2, 3, \ldots \]
\[ u_n(x, t) = B_n \sin \left( \frac{n\pi ct}{L} \right) \sin \left( \frac{n\pi x}{L} \right) \]

The solution is then,

\[ u(x, t) = \sum_{n=1}^{\infty} \left[ A_n \cos \left( \frac{n\pi ct}{L} \right) \sin \left( \frac{n\pi x}{L} \right) + B_n \sin \left( \frac{n\pi ct}{L} \right) \sin \left( \frac{n\pi x}{L} \right) \right] \]

Now, in order to apply the second initial condition we’ll need to differentiate this with respect to \( t \) so,

\[ \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left[ -\frac{n\pi c}{L} A_n \sin \left( \frac{n\pi ct}{L} \right) \sin \left( \frac{n\pi x}{L} \right) + \frac{n\pi c}{L} B_n \cos \left( \frac{n\pi ct}{L} \right) \sin \left( \frac{n\pi x}{L} \right) \right] \]

If we now apply the initial conditions we get,

\[ u(x, 0) = f(x) = \sum_{n=1}^{\infty} \left[ A_n \cos(0) \sin \left( \frac{n\pi x}{L} \right) + B_n \sin(0) \sin \left( \frac{n\pi x}{L} \right) \right] = \sum_{n=1}^{\infty} A_n \sin \left( \frac{n\pi x}{L} \right) \]

\[ \frac{\partial u}{\partial t}(x, 0) = g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin \left( \frac{n\pi x}{L} \right) \]

Both of these are Fourier sine series. The first is for \( f(x) \) on \( 0 \leq x \leq L \) while the second is for \( g(x) \) on \( 0 \leq x \leq L \) with a slightly messy coefficient. As in the last few sections we’re faced with the choice of either using the orthogonality of the sines to derive formulas for \( A_n \) and \( B_n \) or we could reuse formula from previous work.

It’s easier to reuse formulas so using the formulas form the Fourier sine series section we get,
Upon solving the second one we get,

\[ A_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \left( \frac{n \pi x}{L} \right) \, dx \quad n = 1, 2, 3, \ldots \]

\[ \frac{n \pi c}{L} B_n = \frac{2}{L} \int_{0}^{L} g(x) \sin \left( \frac{n \pi x}{L} \right) \, dx \quad n = 1, 2, 3, \ldots \]

So, there is the solution to the 1-D wave equation and with that we’ve solved the final partial differential equation in this chapter.