Introduction

This review was originally written for my Calculus I class but it should be accessible to anyone needing a review in some basic algebra and trig topics. The review contains the occasional comment about how a topic will/can be used in a calculus class. If you aren't in a calculus class you can ignore these comments. I don't cover all the topics that you would see in a typical Algebra or Trig class, I've mostly covered those that I feel would be most useful for a student in a Calculus class although I have included a couple that are not really required for a Calculus class. These extra topics were included simply because they do come up on occasion and I felt like including them. There are also, in all likelihood, a few Algebra/Trig topics that do arise occasionally in a Calculus class that I didn't include.

Because this review was originally written for my Calculus students to use as a test of their algebra and/or trig skills it is generally in the form of a problem set. The solution to the first problem in a set contains detailed information on how to solve that particular type of problem. The remaining solutions are also fairly detailed and may contain further required information that wasn't given in the first problem, but they probably won't contain explicit instructions or reasons for performing a certain step in the solution process. It was my intention in writing the solutions to make them detailed enough that someone needing to learn a particular topic should be able to pick the topic up from the solutions to the problems. I hope that I've accomplished this.

So, why did I even bother to write this?

The ability to do basic algebra is absolutely vital to successfully passing a calculus class. As you progress through a calculus class you will see that almost every calculus problem involves a fair amount of algebra. In fact, in many calculus problems, 90% or more of the problem is algebra.

So, while you may understand the basic calculus concepts, if you can't do the algebra you won't be able to do the problems. If you can't do the problems you will find it very difficult to pass the course.

Likewise you will find that many topics in a calculus class require you to be able to basic trigonometry. In quite a few problems you will be asked to work with trig functions, evaluate trig functions and solve trig equations. Without the ability to do basic trig you will have a hard time doing these problems.

Good algebra and trig skills will also be required in Calculus II or Calculus III. So, if you don't have good algebra or trig skills you will find it very difficult to complete this sequence of courses.

Most of the following set of problems illustrates the kinds of algebra and trig skills that you will need in order to successfully complete any calculus course here at Lamar University. The algebra and trig in these problems fall into three categories :

- Easier than the typical calculus problem,
- similar to a typical calculus problem, and
- harder than a typical calculus problem.

Which category each problem falls into will depend on the instructor you have. In my calculus course you will find that most of these problems falling into the first two categories.

Depending on your instructor, the last few sections (Inverse Trig Functions through Solving Logarithm Equations) may be covered to one degree or another in your class. However, even if your instructor does cover this material you will find it useful to have gone over these sections. In my course I spend the first couple of days covering the basics of exponential and logarithm functions since I tend to use them on a regular basis.

This problem set is not designed to discourage you, but instead to make sure you have the background that is required in order to pass this course. If you have trouble with the material on this worksheet (especially the Exponents - Solving Trig Equations sections) you will find that you will also have a great deal of trouble passing a calculus course.

Please be aware that this problem set is NOT designed to be a substitute for an algebra or trig course. As I have already mentioned I do not cover all the topics that are typically covered in an Algebra or Trig course. The most of the topics covered here are those that I feel are important topics that you MUST have in order to successfully complete a calculus course (in particular my Calculus course). You may find that there are other algebra or trig skills that are also required for you to be successful in this course that are not covered in this review. You may also find that your instructor will not require all the skills that are listed here on this review.

Here is a brief listing and quick explanation of each topic covered in this review.

Algebra

Exponents – A brief review of the basic exponent properties.
Absolute Value – A couple of quick problems to remind you of how absolute value works.
Radicals – A review of radicals and some of their properties.
Rationalizing – A review of a topic that doesn't always get covered all that well in an algebra class, but is required occasionally in a Calculus class.
Functions – Function notation and function evaluation.
Multiplying Polynomials – A couple of polynomial multiplication problems illustrating common mistakes in a Calculus class.
Factoring – Some basic factoring.

Algebra

Exponents

Simplify each of the following as much as possible.

1. $2x^4y^{-3}x^{-19} + y^{\frac{1}{3}}y^{-\frac{3}{4}}$

Solution

All of these problems make use of one or more of the following properties.

$$p^n p^m = p^{n+m} \qquad \frac{p^n}{p^m} = p^{n-m} = \frac{1}{p^{m-n}}$$

$$\left(p^n\right)^m = p^{nm} \qquad p^0 = 1, \text{ provided } p \neq 0$$

$$\left(pq\right)^n = p^n q^n \qquad \left(\frac{p}{q}\right)^n = \frac{p^n}{q^n}$$

$$p^{-n} = \frac{1}{p^n} \qquad \frac{1}{p^{-n}} = p^n$$

$$\left(\frac{p}{q}\right)^{-n} = \left(\frac{q}{p}\right)^n = \frac{q^n}{p^n}$$

This particular problem only uses the first property.

$$2x^4y^{-3}x^{-19} + y^{\frac{1}{3}}y^{-\frac{3}{4}} = 2x^{4-19}y^{-3} + y^{\frac{1}{3}-\frac{3}{4}} = 2x^{-15}y^{-3} + y^{-\frac{5}{12}}$$

Remember that the y's in the last two terms can't be combined! You can only combine terms that are products or quotients. Also, while this would be an acceptable and often preferable answer in a calculus class an algebra class would probably want you to get rid of the negative exponents as well. In this case your answer would be.

$$2x^4y^{-3}x^{-19} + y^{\frac{1}{3}}y^{-\frac{3}{4}} = 2x^{-15}y^{-3} + y^{-\frac{5}{12}} = \frac{2}{x^{15}y^3} + \frac{1}{y^{\frac{5}{12}}}$$

The 2 will stay in the numerator of the first term because it doesn't have a negative exponent.

2. $x^{\frac{3}{5}}x^2x^{-\frac{1}{2}}$

Solution

$$x^{\frac{3}{5}}x^2x^{-\frac{1}{2}} = x^{\frac{3}{5}+2-\frac{1}{2}} = x^{\frac{6}{10}+\frac{20}{10}-\frac{5}{10}} = x^{\frac{21}{10}}$$

Not much to this solution other than just adding the exponents.

3. $\dfrac{xx^{-\frac{1}{3}}}{2x^5}$

Solution

$$\frac{xx^{-\frac{1}{3}}}{2x^5} = \frac{x^{\frac{2}{3}}}{2x^5} = \frac{x^{\frac{2}{3}}x^{-5}}{2} = \frac{x^{-\frac{13}{3}}}{2} = \frac{1}{2}x^{-\frac{13}{3}}$$

Note that you could also have done the following (probably is easier….).

$$\frac{xx^{-\frac{1}{3}}}{2x^5} = \frac{x^{-\frac{1}{3}}}{2x^4} = \frac{x^{-\frac{1}{3}}x^{-4}}{2} = \frac{x^{-\frac{13}{3}}}{2} = \frac{1}{2}x^{-\frac{13}{3}}$$

In the second case I first canceled an x before doing any simplification.

In both cases the 2 stays in the denominator. Had I wanted the 2 to come up to the numerator with the x I would have used $(2x)^5$ in the denominator. So watch parenthesis!

4. $\left(\frac{2x^{-2}x^{\frac{4}{5}}y^6}{x+y}\right)^{-3}$

Solution
There are a couple of ways to proceed with this problem. I'm going to first simplify the inside of the parenthesis a little. At the same time I'm going to use the last property above to get rid of the minus sign on the whole thing.

$$\left(\frac{2x^{-2}x^{\frac{4}{5}}y^6}{x+y}\right)^{-3} = \left(\frac{x+y}{2x^{-\frac{6}{5}}y^6}\right)^3$$

Now bring the exponent in. Remember that every term (including the 2) needs to get the exponent.

$$\left(\frac{2x^{-2}x^{\frac{4}{5}}y^6}{x+y}\right)^{-3} = \frac{(x+y)^3}{2^3\left(x^{-\frac{6}{5}}\right)^3\left(y^6\right)^3} = \frac{(x+y)^3}{8x^{-\frac{18}{5}}y^{18}}$$

Recall that $(x+y)^3 \neq x^3 + y^3$ so you can't go any further with this.

5. $\left(\frac{x^{\frac{4}{7}}x^{\frac{9}{2}}x^{-\frac{10}{3}} - x^2x^{-9}x^{\frac{1}{2}}}{x+1}\right)^0$

Solution
Don't make this one harder than it has to be. Note that the whole thing is raised to the zero power so there is only one property that needs to be used here.

$$\left(\frac{x^{\frac{4}{7}}x^{\frac{9}{2}}x^{-\frac{10}{3}} - x^2x^{-9}x^{\frac{1}{2}}}{x+1}\right)^0 = 1$$

Absolute Value

1. Evaluate $|5|$ and $|-123|$

   Solution

   To do these evaluations we need to remember the definition of absolute value.

$$|p| = \begin{cases} p & \text{if } p \ge 0 \\ -p & \text{if } p < 0 \end{cases}$$

With this definition the evaluations are easy.

$$|5| = 5 \qquad \text{because } 5 \ge 0$$

$$|-123| = -(-123) = 123 \qquad \text{because } -123 < 0$$

   Remember that absolute value takes any nonzero number and makes sure that it's positive.

2. Eliminate the absolute value bars from $|3-8x|$

   Solution

   This one is a little different from the first example. We first need to address a very common mistake with these.

$$|3-8x| \ne 3+8x$$

   Absolute value doesn't just change all minus signs to plus signs. Remember that absolute value takes a number and makes sure that it's positive or zero. To convince yourself of this try plugging in a number, say $x = -10$

$$83 = |83| = |3+80| = |3-8(-10)| \ne 3+8(-10) = 3-80 = -77$$

   There are two things wrong with this. First, is the fact that the two numbers aren't even close to being the same so clearly it can't be correct. Also note that if absolute value is supposed to make nonzero numbers positive how can it be that we got a -77 of out of it? Either one of these should show you that this isn't correct, but together they show real problems with doing this, so don't do it!

   That doesn't mean that we can't eliminate the absolute value bars however. We just need to figure out what values of *x* will give positive numbers and what values of *x* will give negative numbers. Once we know this we can eliminate the absolute value bars. First notice the following (you do remember how to Solve Inequalities right?)

$$3-8x \ge 0 \quad \Rightarrow \quad 3 \ge 8x \quad \Rightarrow \quad x \le \frac{3}{8}$$

$$3-8x < 0 \quad \Rightarrow \quad 3 < 8x \quad \Rightarrow \quad x > \frac{3}{8}$$

   So, if $x \le \frac{3}{8}$ then $3-8x \ge 0$ and if $x > \frac{3}{8}$ then $3-8x < 0$. With this information we can now eliminate the absolute value bars.

$$|3-8x| = \begin{cases} 3-8x & \text{if } x \le \frac{3}{8} \\ -(3-8x) & \text{if } x > \frac{3}{8} \end{cases}$$

Or,

$$|3-8x| = \begin{cases} 3-8x & \text{if } x \le \frac{3}{8} \\ -3+8x & \text{if } x > \frac{3}{8} \end{cases}$$

So, we can still eliminate the absolute value bars but we end up with two different formulas and the formula that we will use will depend upon what value of x that we've got.

On occasion you will be asked to do this kind of thing in a calculus class so it's important that you can do this when the time comes around.

3. List as many of the properties of absolute value as you can.

Solution

Here are a couple of basic properties of absolute value.

$$|p| \ge 0 \qquad\qquad |-p| = |p|$$

These should make some sense. The first is simply restating the results of the definition of absolute value. In other words, absolute value makes sure the result is positive or zero (if $p = 0$). The second is also a result of the definition. Since taking absolute value results in a positive quantity or zero it won't matter if there is a minus sign in there or not.

We can use absolute value with products and quotients as follows

$$|ab| = |a|\,|b| \qquad\qquad \left|\frac{a}{b}\right| = \frac{|a|}{|b|}$$

Notice that I didn't include sums (or differences) here. That is because in general

$$|a+b| \ne |a| + |b|$$

To convince yourself of this consider the following example

$$7 = |-7| = |2-9| = |2+(-9)| \ne |2| + |-9| = 2+9 = 11$$

Clearly the two aren't equal. This does lead to something that is often called the triangle inequality. The triangle inequality is

$$|a+b| \le |a| + |b|$$

The triangle inequality isn't used all that often in a Calculus course, but it's a nice property of absolute value so I thought I'd include it.

Radicals

Evaluate the following.

1. $\sqrt[3]{125}$

Solution
In order to evaluate radicals all that you need to remember is

$$y = \sqrt[n]{x} \text{ is equivalent to } x = y^n$$

In other words, when evaluating $\sqrt[n]{x}$ we are looking for the value, *y*, that we raise to the *n* to get *x*. So, for this problem we've got the following.

$\sqrt[3]{125} = 5$ because $5^3 = 125$

2. $\sqrt[6]{64}$

Solution
$\sqrt[6]{64} = 2$ because $2^6 = 64$

3. $\sqrt[5]{-243}$

Solution
$\sqrt[5]{-243} = -3$ because $(-3)^5 = -243$

4. $\sqrt[2]{100}$

Solution
$\sqrt[2]{100} = \sqrt{100} = 10$ Remember that $\sqrt[2]{x} = \sqrt{x}$

5. $\sqrt[4]{-16}$

Solution
$\sqrt[4]{-16} = \text{n/a}$
Technically, the answer to this problem is a complex number, but in most calculus classes, including mine, complex numbers are not dealt with. There is also the fact that it's beyond the scope of this review to go into the details of getting a complex answer to this problem.

Convert each of the following to exponential form.

6. $\sqrt{7x}$

Solution

To convert radicals to exponential form you need to remember the following formula

$$\sqrt[n]{p} = p^{\frac{1}{n}}$$

For this problem we've got.

$$\sqrt{7x} = \sqrt[2]{7x} = (7x)^{\frac{1}{2}}$$

There are a couple of things to note with this one. Remember $\sqrt{p} = \sqrt[2]{p}$ and notice the parenthesis. These are required since both the 7 and the *x* was under the radical so both must also be raised to the power. The biggest mistake made here is to convert this as

$$7x^{\frac{1}{2}}$$

however this is incorrect because

$$7x^{\frac{1}{2}} = 7\sqrt{x}$$

Again, be careful with the parenthesis

7. $\sqrt[5]{x^2}$

Solution

$\sqrt[5]{x^2} = \left(x^2\right)^{\frac{1}{5}} = x^{\frac{2}{5}}$

Note that I combined exponents here. You will always want to do this.

8. $\sqrt[3]{4x+8}$

Solution

$\sqrt[3]{4x+8} = (4x+8)^{\frac{1}{3}}$

You CANNOT simplify further to $(4x)^{\frac{1}{3}} + (8)^{\frac{1}{3}}$ so don't do that!!!! Remember that $(a+b)^n \neq a^n + b^n$!!!!

Simplify each of the following.

9. $\sqrt[3]{16x^6y^{13}}$ Assume that $x \geq 0$ and $y \geq 0$ for this problem.

Solution

The property to use here is

$$\sqrt[n]{xy} = \sqrt[n]{x}\ \sqrt[n]{y}$$

A similar property for quotients is

$$\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

Both of these properties require that at least one of the following is true $x \geq 0$ and/or $y \geq 0$. To see why this is the case consider the following example

$$4 = \sqrt{16} = \sqrt{(-4)(-4)} \neq \sqrt{-4}\sqrt{-4} = (2i)(2i) = 4i^2 = -4$$

If we try to use the property when both are negative numbers we get an incorrect answer. If you don't know or recall complex numbers you can ignore this example.

The property will hold if one is negative and the other is positive, but you can't have both negative.

I'll also need the following property for this problem.

$$\sqrt[n]{x^n} = x \qquad \text{provided } n \text{ is odd}$$

In the next example I'll deal with *n* even.

Now, on to the solution to this example. I'll first rewrite the stuff under the radical a little then use both of the properties that I've given here.

$$\begin{aligned}\sqrt[3]{16x^6y^{13}} &= \sqrt[3]{8x^3x^3\ y^3y^3y^3y^3\ 2y} \\ &= \sqrt[3]{8}\sqrt[3]{x^3}\sqrt[3]{x^3}\sqrt[3]{y^3}\sqrt[3]{y^3}\sqrt[3]{y^3}\sqrt[3]{y^3}\sqrt[3]{2y} \\ &= 2x\,x\,y\,y\,y\,y\sqrt[3]{2y} \\ &= 2x^2y^4\sqrt[3]{2y}\end{aligned}$$

So, all that I did was break up everything into terms that are perfect cubes and terms that weren't perfect cubes. I then used the property that allowed me to break up a product under the radical. Once this was done I simplified each perfect cube and did a little combining.

10. $\sqrt[4]{16x^8y^{15}}$

Solution

I did not include the restriction that $x \geq 0$ and $y \geq 0$ in this problem so we're going to have to be a little more careful here. To do this problem we will need the following property.

$$\sqrt[n]{x^n} = |x| \qquad \text{provided } n \text{ is even}$$

To see why the absolute values are required consider $\sqrt{4}$. When evaluating this we are really asking what number did we square to get four? The problem is there are in fact two answer to this : 2 and -2! When evaluating square roots (or any even root for that matter) we want a predicable answer. We don't want to have to sit down each and every time and decide whether we want the positive or negative number. Therefore, by putting the absolute value bars on the *x* we will guarantee that the answer is always positive and hence predictable.

So, what do we do if we know that we want the negative number? Simple. We add a minus sign in front of the square root as follows

$$-\sqrt{4} = -\left(2\right) = -2$$

This gives the negative number that we wanted and doesn't violate the rule that square root always return the positive number!

Okay, let's finally do this problem. For the most part it works the same as the previous one did, we just have to be careful with the absolute value bars.

$$\begin{aligned}\sqrt[4]{16x^8y^{15}} &= \sqrt[4]{16x^4x^4\ y^4y^4y^4\ y^3}\\ &= 2\left|x\right|\left|x\right|\left|y\right|\left|y\right|\left|y\right|\sqrt[4]{y^3}\\ &= 2\left|x\right|^2\left|y\right|^3\sqrt[4]{y^3}\\ &= 2x^2\left|y\right|^3\sqrt[4]{y^3}\end{aligned}$$

Note that I could drop the absolute value on the $\left|x^2\right|$ term because the power of 2 will give a positive answer for x^2 regardless of the sign of *x*. They do need to stay on the *y* term however because of the power.

Rationalizing

Rationalize each of the following.

1. $\dfrac{3xy}{\sqrt{x}+\sqrt{y}}$

Solution

This is the typical rationalization problem that you will see in an algebra class. In these kinds of problems you want to eliminate the square roots from the denominator. To do this we will use

$$\left(a+b\right)\left(a-b\right) = a^2 - b^2 .$$

So, to rationalize the denominator (in this case, as opposed to the next problem) we will multiply the numerator and denominator by $\sqrt{x}-\sqrt{y}$. Remember, that to rationalize we simply multiply numerator and denominator by the term containing the roots with the sign between them changed. So, in this case, we had $\sqrt{x}+\sqrt{y}$ and so we needed to change the "+" to a "-".

Now, back to the problem. Here's the multiplication.

$$\frac{3xy}{\left(\sqrt{x}+\sqrt{y}\right)}\ \frac{\left(\sqrt{x}-\sqrt{y}\right)}{\left(\sqrt{x}-\sqrt{y}\right)} = \frac{3xy\left(\sqrt{x}-\sqrt{y}\right)}{x-y}$$

Note that the results will often be "messier" than the original expression. However, as you will see in your calculus class there are certain problems that can only be easily worked if the problem has first been rationalized.

Unfortunately, sometimes you have to make the problem more complicated in order to work with it.

2. $\dfrac{\sqrt{t+2}-2}{t^2-4}$

Solution
In this problem we're going to rationalize the numerator. Do NOT get too locked into always rationalizing the denominator. You will need to be able to rationalize the numerator occasionally in a calculus class. It works in pretty much the same way however.

$$\begin{aligned}\frac{\left(\sqrt{t+2}-2\right)}{\left(t^2-4\right)}\frac{\left(\sqrt{t+2}+2\right)}{\left(\sqrt{t+2}+2\right)} &= \frac{t+2-4}{\left(t^2-4\right)\left(\sqrt{t+2}+2\right)} \\ &= \frac{t-2}{\left(t-2\right)\left(t+2\right)\left(\sqrt{t+2}+2\right)} \\ &= \frac{1}{\left(t+2\right)\left(\sqrt{t+2}+2\right)}\end{aligned}$$

Notice that, in this case there was some simplification we could do after the rationalization. This will happen occasionally.

Functions

1. Given $f(x)=-x^2+6x-11$ and $g(x)=\sqrt{4x-3}$ find each of the following.

(a) $f(2)$ (b) $g(2)$ (c) $f(-3)$ (d) $g(10)$
(e) $f(t)$ (f) $f(t-3)$ (g) $f(x-3)$ (h) $f(4x-1)$

Solution
All throughout a calculus sequence you will be asked to deal with functions so make sure that you are familiar and comfortable with the notation and can evaluate functions.

First recall that the $f(x)$ in a function is nothing more than a fancy way of writing the y in an equation so

$$f(x) = -x^2 + 6x - 11$$

is equivalent to writing

$$y = -x^2 + 6x - 11$$

except the function notation form, while messier to write, is much more convenient for the types of problem you'll be working in a Calculus class.

In this problem we're asked to evaluate some functions. So, in the first case $f(2)$ is asking us to determine the value of $y = -x^2 + 6x - 11$ when $x = 2$.

The key to remembering how to evaluate functions is to remember that you whatever is in the parenthesis on the left is substituted in for all the *x*'s on the right side.

So, here are the function evaluations.

(a) $f(2) = -(2)^2 + 6(2) - 11 = -3$

(b) $g(2) = \sqrt{4(2) - 3} = \sqrt{5}$

(c) $f(-3) = -(-3)^2 + 6(-3) - 11 = -38$

(d) $g(10) = \sqrt{4(10) - 3} = \sqrt{37}$

(e) $f(t) = -t^2 + 6t - 11$

Remember that we substitute for the *x*'s WHATEVER is in the parenthesis on the left. Often this will be something other than a number. So, in this case we put *t*'s in for all the *x*'s on the left.

This is the same as we did for **(a) – (d)** except we are now substituting in something other than a number. Evaluation works the same regardless of whether we are substituting a number or something more complicated.

(f) $f(t-3) = -(t-3)^2 + 6(t-3) - 11 = -t^2 + 12t - 38$

Often instead of evaluating functions at numbers or single letters we will have some fairly complex evaluations so make sure that you can do these kinds of evaluations.

(g) $f(x-3) = -(x-3)^2 + 6(x-3) - 11 = -x^2 + 12x - 38$

The only difference between this one and the previous one is that I changed the *t* to an *x*. Other than that there is absolutely no difference between the two!

Do not let the fact that there are *x*'s in the parenthesis on the left get you worked up! Simply replace all the *x*'s in the formula on the right side with $x-3$. This one works exactly the same as**(f).**

(h) $f(4x-1) = -(4x-1)^2 + 6(4x-1) - 11 = -16x^2 + 32x - 18$

Do not get excited by problems like **(e) – (h)**. This type of problem works the same as **(a) – (d)** we just aren't using numbers! Instead of substituting numbers you are substituting letters and/or other functions. So, if you can do **(a) – (d)** you can do these more complex function evaluations as well!

2. Given $f(x) = 10$ find each of the following.

(a) $f(7)$ (b) $f(0)$ (c) $f(-14)$

Solution

This is one of the simplest functions in the world to evaluate, but for some reason seems to cause no end of difficulty for students. Recall from the previous problem how function evaluation works. We replace every *x* on the right side with what ever is in the parenthesis on the left. However, in this case since there are no *x*'s on the right side (this is probably what causes the problems) we simply get 10 out of each of the function evaluations. This kind of function is called a **constant function**. Just to be clear here are the function evaluations.

$$f(7) = 10 \qquad f(0) = 10 \qquad f(-14) = 10$$

3. Given $f(x) = 3x^2 - x + 10$ and $g(x) = 1 - 20x$ find each of the following.

(a) $(f-g)(x)$ (b) $\left(\frac{f}{g}\right)(x)$ (c) $(fg)(x)$

(d) $(f \circ g)(5)$ (e) $(f \circ g)(x)$ (f) $(g \circ f)(x)$

Solution

This problem makes sure you are familiar with notation commonly used with functions. The appropriate formulas are included in the answer to each part.

(a)

$$\begin{aligned}(f-g)(x) &= f(x) - g(x) \\ &= 3x^2 - x + 10 - (1 - 20x) \\ &= 3x^2 - x + 10 - 1 + 20x \\ &= 3x^2 + 19x + 9\end{aligned}$$

(b)

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$
$$= \frac{3x^2 - x + 10}{1 - 20x}$$

(c)
$$(fg)(x) = f(x)\,g(x)$$
$$= \left(3x^2 - x + 10\right)(1 - 20x)$$
$$= -60x^3 + 23x^2 - 201x + 10$$

(d) For this problems (and the next two) remember that the little circle , $\circ$, in this problem signifies that we are doing composition NOT multiplication!

The basic formula for composition is
$$(f \circ g)(x) = f\left(g(x)\right)$$
In other words, you plug the second function listed into the first function listed then evaluate as appropriate.

In this case we've got a number instead of an *x* but it works in exactly the same way.
$$(f \circ g)(5) = f\left(g(5)\right)$$
$$= f(-99)$$
$$= 29512$$

(e) Compare the results of this problem to **(c)**! Composition is NOT the same as multiplication so be careful to not confuse the two!
$$(f \circ g)(x) = f\left(g(x)\right)$$
$$= f(1 - 20x)$$
$$= 3(1 - 20x)^2 - (1 - 20x) + 10$$
$$= 3\left(1 - 40x + 400x^2\right) - 1 + 20x + 10$$
$$= 1200x^2 - 100x + 12$$

(f) Compare the results of this to **(e)**! The order in which composition is written is important! Make sure you pay attention to the order.
$$(g \circ f)(x) = g\left(f(x)\right)$$
$$= g\left(3x^2 - x + 10\right)$$
$$= 1 - 20\left(3x^2 - x + 10\right)$$
$$= -60x^2 + 20x - 199$$

Multiplying Polynomials

Multiply each of the following.

1. $\left(7x-4\right)\left(7x+4\right)$

Solution

Most people remember learning the FOIL method of multiplying polynomials from an Algebra class. I'm not very fond of the FOIL method for the simple reason that it only works when you are multiplying two polynomials each of which has exactly two terms (*i.e.* you're multiplying two binomials). If you have more than two polynomials or either of them has more or less that two terms in it the FOIL method fails.

The FOIL method has its purpose, but you've got to remember that it doesn't always work. The correct way to think about multiplying polynomials is to remember the rule that every term in the second polynomial gets multiplied by every term in the first polynomial.

So, in this case we've got.

$$\left(7x-4\right)\left(7x+4\right)=49x^2+28x-28x-16=49x^2-16$$

Always remember to simplify the results if possible and combine like terms.

This problem was to remind you of the formula

$$\left(a+b\right)\left(a-b\right)=a^2-b^2$$

2. $\left(2x-5\right)^2$

Solution

Remember that $3^2=\left(3\right)\left(3\right)$ and so

$$\left(2x-5\right)^2=\left(2x-5\right)\left(2x-5\right)=4x^2-20x+25$$

This problem is to remind you that

$$\left(a-b\right)^n\neq a^n-b^n \qquad \text{and} \qquad \left(a+b\right)^n\neq a^n+b^n$$

so do not make that mistake!

$$\left(2x-5\right)^2\neq 4x^2-25$$

There are actually a couple of formulas here.

$$\left(a+b\right)^2=a^2+2ab+b^2$$

$$\left(a-b\right)^2=a^2-2ab+b^2$$

You can memorize these if you'd like, but if you don't remember them you can always just FOIL out the two polynomials and be done with it…

3. $2\left(x+3\right)^2$

Solution
Be careful in dealing with the 2 out in front of everything. Remember that order of operations tells us that we first need to square things out before multiplying the 2 through.

$$2\left(x+3\right)^2 = 2\left(x^2+6x+9\right) = 2x^2+12x+18$$

Do, do not do the following

$$2\left(x+3\right)^2 \neq \left(2x+6\right)^2 = 4x^2+24x+36$$

It is clear that if you multiply the 2 through before squaring the term out you will get very different answers!

There is a simple rule to remember here. You can only distribute a number through a set of parenthesis if there isn't any exponent on the term in the parenthesis.

4. $\left(2x^3-x\right)\left(\sqrt{x}+\frac{2}{x}\right)$

Solution
While the second term is not a polynomial you do the multiplication in exactly same way. The only thing that you've got to do is first convert everything to exponents then multiply.

$$\begin{aligned}\left(2x^3-x\right)\left(\sqrt{x}+\frac{2}{x}\right) &= \left(2x^3-x\right)\left(x^{\frac{1}{2}}+2x^{-1}\right) \\ &= 4x^2+2x^{\frac{7}{2}}-x^{\frac{3}{2}}-2\end{aligned}$$

5. $\left(3x+2\right)\left(x^2-9x+12\right)$

Solution
Remember that the FOIL method will not work on this problem. Just multiply every term in the second polynomial by every term in the first polynomial and you'll be done.

$$\begin{aligned}\left(3x+2\right)\left(x^2-9x+12\right) &= x^2\left(3x+2\right)-9x\left(3x+2\right)+12\left(3x+2\right) \\ &= 3x^3-25x^2+18x+24\end{aligned}$$

Factoring

Factor each of the following as much as possible.

1. $100x^2 - 81$

Solution

We have a difference of squares and remember not to make the following mistake.

$$100x^2 - 81 \neq (10x - 9)^2$$

This just simply isn't correct. To convince yourself of this go back to Problems 1 and 2 in the Multiplying Polynomials section. Here is the correct answer.

$$100x^2 - 81 = (10x - 9)(10x + 9)$$

2. $100x^2 + 81$

Solution

This is a sum of squares and a sum of squares can't be factored, except in rare cases, so this is as factored as it will get. As noted there are some rare cases in which a sum of squares can be factored but you will, in all likelihood never run into one of them.

3. $3x^2 + 13x - 10$

Solution

Factoring this kind of polynomial is often called trial and error. It will factor as

$$(ax + b)(cx + d)$$

where $ac = 3$ and $bd = -10$. So, you find all factors of 3 and all factors of -10 and try them in different combinations until you get one that works. Once you do enough of these you'll get to the point that you can usually get them correct on the first or second guess. The only way to get good at these is to just do lot's of problems.

Here's the answer for this one.

$$3x^2 + 13x - 10 = (3x - 2)(x + 5)$$

4. $25x^2 + 10x + 1$

Solution

There's not a lot to this problem.

$$25x^2 + 10x + 1 = (5x + 1)(5x + 1) = (5x + 1)^2$$

When you run across something that turns out to be a perfect square it's usually best write it as such.

5. $4x^5 - 8x^4 - 32x^3$

Solution
In this case don't forget to always factor out any common factors first before going any further.

$$4x^5 - 8x^4 - 32x^3 = 4x^3\left(x^2 - 2x - 8\right) = 4x^3\left(x-4\right)\left(x+2\right)$$

6. $125x^3 - 8$

Solution
Remember the basic formulas for factoring a sum or difference of cubes.

$$a^3 - b^3 = \left(a-b\right)\left(a^2 + ab + b^2\right)$$
$$a^3 + b^3 = \left(a+b\right)\left(a^2 - ab + b^2\right)$$

In this case we've got

$$125x^3 - 8 = \left(5x-2\right)\left(25x^2 + 10x + 4\right)$$

Simplifying Rational Expressions

Simplify each of the following rational expressions.

1. $\dfrac{2x^2 - 8}{x^2 - 4x + 4}$

Solution
There isn't a lot to these problems. Just factor the numerator and denominator as much as possible then cancel all like terms. Recall that you can only cancel terms that multiply the whole numerator and whole denominator. In other words, you can't just cancel the x^2 and you can't cancel a 4 from the 8 in the numerator and the 4 in the denominator. Things just don't work this way.

If you need convincing of this consider the following number example.

$$8.5 = \frac{17}{2} = \frac{8+9}{2} \neq \frac{4+9}{1} = 13$$

Here is the answer to this problem.

$$\frac{2x^2 - 8}{x^2 - 4x + 4} = \frac{2\left(x^2 - 4\right)}{x^2 - 4x + 4} = \frac{2\left(x-2\right)\left(x+2\right)}{\left(x-2\right)^2} = \frac{2\left(x+2\right)}{x-2}$$

2. $\dfrac{x^2 - 5x - 6}{6x - x^2}$

Solution

In this one you've got to be a little careful. First factor the numerator and denominator.

$$\frac{x^2-5x-6}{6x-x^2}=\frac{(x-6)(x+1)}{x(6-x)}$$

At first glance it doesn't look like anything will cancel. However remember that

$$a-b=-(b-a)$$

Using this on the term in the denominator gives the following.

$$\frac{x^2-5x-6}{6x-x^2}=\frac{(x-6)(x+1)}{x(6-x)}=\frac{(x-6)(x+1)}{-x(x-6)}=\frac{x+1}{-x}=-\frac{x+1}{x}$$

Also recall that

$$\frac{a}{-b}=\frac{-a}{b}=-\frac{a}{b}$$

so it doesn't matter where you put the minus sign.

Graphing and Common Graphs

Sketch the graph of each of the following.

1. $y=-\frac{2}{5}x+3$

Solution
This is a line in the slope intercept form

$$y=mx+b$$

In this case the line has a *y* intercept of (0,*b*) and a slope of *m*. Recall that slope can be thought of as

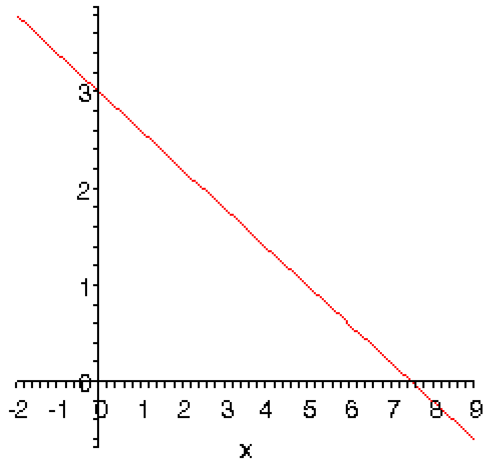
$$m=\frac{\text{rise}}{\text{run}}$$

If the slope is negative we tend to think of the rise as a fall.

The slope allows us to get a second point on the line. Once we have any point on the line and the slope we move right by *run* and up/down by *rise* depending on the sign. This will be a second point on the line.

In this case we know (0,3) is a point on the line and the slope is $-\frac{2}{5}$. So starting at (0,3) we'll move 5 to the right (*i.e.* $0\rightarrow 5$) and down 2 (*i.e.* $3\rightarrow 1$) to get (5,1) as a second point on the line. Once we've got two points on a line all we need to do is plot the two points and connect them with a line.

Here's the sketch for this line.



2. $y=(x+3)^2-1$

Solution
This is a parabola in the form

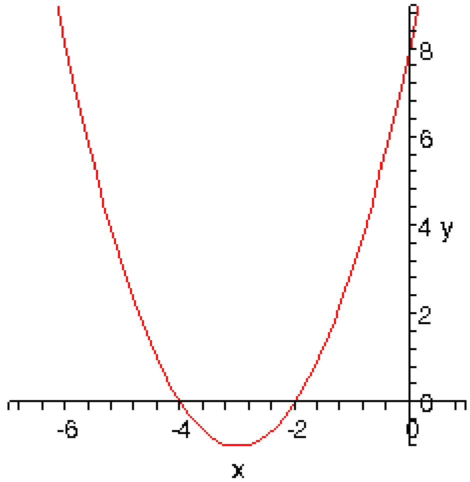
$$y=(x-h)^2+k$$

Parabolas in this form will have the vertex at (*h, k*). So, the vertex for our parabola is (-3,-1). We can also notice that this parabola will open up since the coefficient of the squared term is positive (and of course it would open down if it was negative).

In graphing parabolas it is also convenient to have the *x*-intercepts, if they exist. In this case they will exist since the vertex is below the *x*-axis and the parabola opens up. To find them all we need to do is solve.

$$(x+3)^2-1=0$$
$$(x+3)^2=1$$
$$x+3=\pm 1$$
$$x=-4 \text{ and } x=-2$$

With this information we can plot the main points and sketch in the parabola. Here is the sketch.



3. $y = -x^2 + 2x + 3$

Solution

This is also the graph of a parabola only it is in the more general form.

$$y = ax^2 + bx + c$$

In this form, the *x*-coordinate of the vertex is $x = -\frac{b}{2a}$ and we get the *y*-coordinate by plugging this value back into the equation. So, for our parabola the coordinates of the vertex will be.

$$x = -\frac{2}{2(-1)} = 1$$

$$y = -(1)^2 + 2(1) + 3 = 4$$

So, the vertex for this parabola is (1,4).

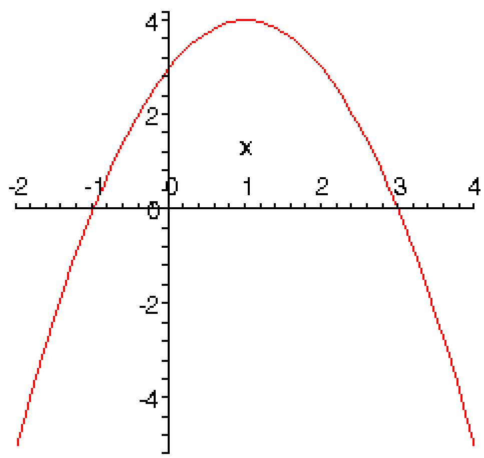
We can also determine which direction the parabola opens from the sign of *a*. If *a* is positive the parabola opens up and if *a* is negative the parabola opens down. In our case the parabola opens down.

This also means that we'll have *x*-intercepts on this graph. So, we'll solve the following.

$$\begin{aligned} -x^2 + 2x + 3 &= 0 \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0 \end{aligned}$$

So, we will have *x*-intercepts at $x = -1$ and $x = 3$. Notice that to make my life easier in the solution process I multiplied everything by -1 to get the coefficient of the x^2 positive. This made the factoring easier.

Here's a sketch of this parabola.



We could also use completing the square to convert this into the form used in the previous problem. To do this I'll first factor out a -1 since it's always easier to complete the square if the coefficient of the x^2 is a positive 1.

$$\begin{aligned} y &= -x^2 + 2x + 3 \\ &= -\left(x^2 - 2x - 3\right) \end{aligned}$$

Now take half the coefficient of the *x* and square that. Then add and subtract this to the quantity inside the parenthesis. This will make the first three terms a perfect square which can be factored.

$$\begin{aligned} y &= -\left(x^2 - 2x - 3\right) \\ &= -\left(x^2 - 2x + 1 - 1 - 3\right) \\ &= -\left(x^2 - 2x + 1 - 4\right) \\ &= -\left(\left(x-1\right)^2 - 4\right) \end{aligned}$$

The final step is to multiply the minus sign back through.

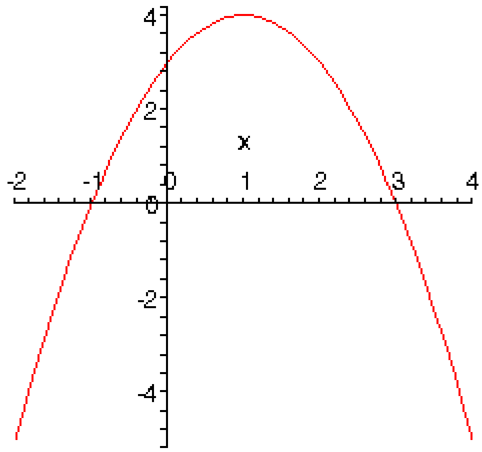
$$y = -\left(x-1\right)^2 + 4$$

4. $f\left(x\right) = -x^2 + 2x + 3$

Solution

The only difference between this problem and the previous problem is that we've got an $f\left(x\right)$ on the left instead of a *y*. However, remember that $f\left(x\right)$ is just a fancy way of writing the *y* and so graphing $f\left(x\right) = -x^2 + 2x + 3$ is identical to graphing $y = -x^2 + 2x + 3$.

So, see the previous problem for the work. For completeness sake here is the graph.



5. $x = y^2 - 6y + 5$

Solution
Most people come out of an Algebra class capable of dealing with functions in the form $y = f(x)$. However, many functions that you will have to deal with in a Calculus class are in the form $x = f(y)$ and can only be easily worked with in that form. So, you need to get used to working with functions in this form.

The nice thing about these kinds of function is that if you can deal with functions in the form $y = f(x)$ then you can deal with functions in the form $x = f(y)$.

So,

$$y = x^2 - 6x + 5$$

is a parabola that opens up and has a vertex of (3,-4).

Well our function is in the form

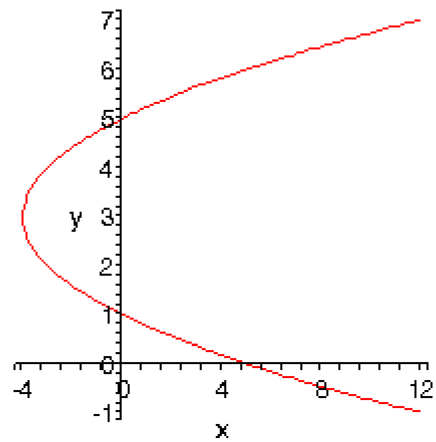
$$x = ay^2 + by + c$$

and this is a parabola that opens left or right depending on the sign of *a* (right if *a* is positive and left if *a* is negative). The *y*-coordinate of the vertex is given by $y = -\frac{b}{2a}$ and we find the *x*-coordinate by plugging this into the equation.

Our function is a parabola that opens to the right (*a* is positive) and has a vertex at (-4,3). To graph this we'll need *y*-intercepts. We find these just like we found *x*-intercepts in the previous couple of problems.

$$y^2 - 6y + 5 = 0$$
$$(y-5)(y-1) = 0$$

The parabola will have *y*-intercepts at $y = 1$ and $y = 5$. Here's a sketch of the graph.



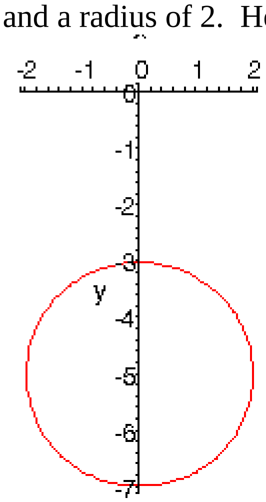
6. $x^2 + (y+5)^2 = 4$

Solution
This is a circle in it s standard form

$$(x-h)^2 + (y-k)^2 = r^2$$

When circles are in this form we can easily identify the center : (h, k) and radius : r. Once we have these we can graph the circle simply by starting at the center and moving right, left, up and down by r to get the rightmost, leftmost, top most and bottom most points respectively.

Our circle has a center at (0, -5) and a radius of 2. Here's a sketch of this circle.



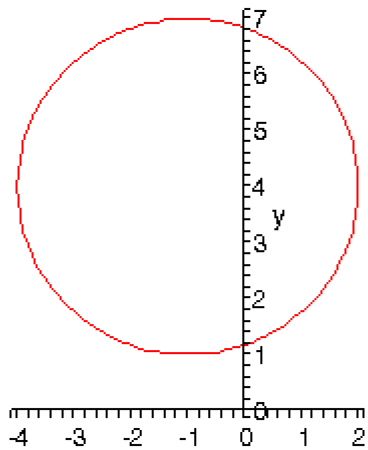
7. $x^2 + 2x + y^2 - 8y + 8 = 0$

Solution

This is also a circle, but it isn't in standard form. To graph this we'll need to do it put it into standard form. That's easy enough to do however. All we need to do is complete the square on the x's and on the y's.

$$x^2+2x+y^2-8y+8=0$$
$$x^2+2x+1-1+y^2-8y+16-16+8=0$$
$$(x+1)^2+(y-4)^2=9$$

So, it looks like the center is (-1, 4) and the radius is 3. Here's a sketch of the circle.



8. $\dfrac{(x-2)^2}{9}+4(y+2)^2=1$

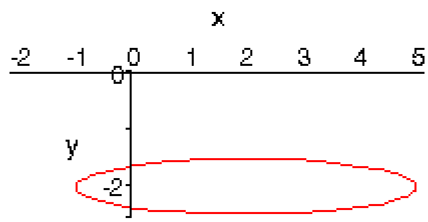
Solution

This is an ellipse. The standard form of the ellipse is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

This is an ellipse with center (h, k) and the right most and left most points are a distance of a away from the center and the top most and bottom most points are a distance of b away from the center.

The ellipse for this problem has center (2, -2) and has $a=3$ and $b=\frac{1}{2}$. Here's a sketch of the ellipse.



9. $\dfrac{(x+1)^2}{9} - \dfrac{(y-2)^2}{4} = 1$

Solution
This is a hyperbola. There are actually two standard forms for a hyperbola.

Form	$\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$	$\dfrac{(y-h)^2}{b^2} - \dfrac{(x-k)^2}{a^2} = 1$
Center	(h, k)	(k, h)
Opens	Opens right and left	Opens up and down
Vertices	a units right and left of center.	b units up and down from center.
Slope of Asymptotes	$\pm\dfrac{b}{a}$	$\pm\dfrac{b}{a}$

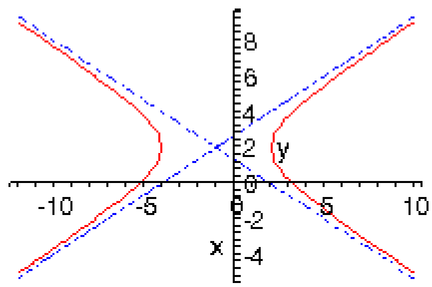
So, what does all this mean? First, notice that one of the terms is positive and the other is negative. This will determine which direction the two parts of the hyperbola open. If the x term is positive the hyperbola opens left and right. Likewise, if the y term is positive the hyperbola opens up and down.

Both have the same "center". Hyperbolas don't really have a center in the sense that circles and ellipses have centers. The center is the starting point in graphing a hyperbola. It tells up how to get to the vertices and how to get the asymptotes set up.

The asymptotes of a hyperbola are two lines that intersect at the center and have the slopes listed above. As you move farther out from the center the graph will get closer and closer to they asymptotes.

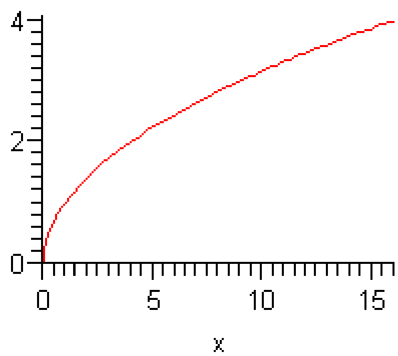
For the equation listed here the hyperbola will open left and right. Its center is (-1, 2). The two vertices are (-4, 2) and (2, 2). The asymptotes will have slopes $\pm\frac{2}{3}$.

Here is a sketch of this hyperbola.



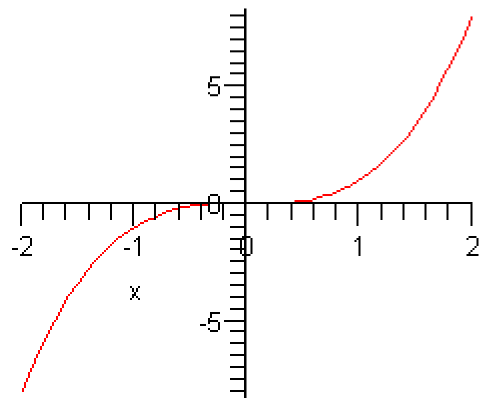
10. $y = \sqrt{x}$

Solution
There isn't much to this problem. This is just the square root and it's a graph that you need to be able to sketch on occasion so here it is. Just remember that you can't plug any negative *x*'s into the square root. Here is the graph.



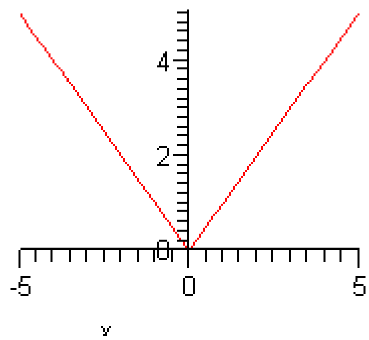
11. $y = x^3$

Solution
Another simple graph that doesn't really need any discussion, it is here simply to make sure you can sketch it.



12. $y = |x|$

Solution
This last sketch is not that difficult if you remember how to evaluate Absolute Value functions. Here is the sketch for this one.



Solving Equations, Part I

Solve each of the following equations.

1. $x^3 - 3x^2 = x^2 + 21x$

Solution
To solve this equation we'll just get everything on side of the equation, factor then use the fact that if $ab = 0$ then either $a = 0$ or $b = 0$.

$$x^3 - 3x^2 = x^2 + 21x$$
$$x^3 - 4x^2 - 21x = 0$$
$$x\left(x^2 - 4x - 21\right) = 0$$
$$x\left(x - 7\right)\left(x + 3\right) = 0$$

So, the solutions are $x = 0$, $x = 7$, and $x = -3$.

Remember that you are being asked to solve this not simplify it! Therefore, make sure that you don't just cancel an x out of both sides! If you cancel an x out as this will cause you to miss $x=0$ as one of the solutions! This is one of the more common mistakes that people make in solving equations.

2. $3x^2-16x+1=0$

Solution
In this case the equation won't factor so we'll need to resort to the quadratic formula. Recall that if we have a quadratic in standard form,

$$ax^2+bx+c=0$$

the solution is,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

So, the solution to this equation is

$$\begin{aligned} x &= \frac{-(-16)\pm\sqrt{(-16)^2-4(3)(1)}}{2(3)} \\ &= \frac{16\pm\sqrt{244}}{6} \\ &= \frac{16\pm 2\sqrt{61}}{6} \\ &= \frac{8\pm\sqrt{61}}{3} \end{aligned}$$

Do not forget about the quadratic formula! Many of the problems that you'll be asked to work in a Calculus class don't require it to make the work go a little easier, but you will run across it often enough that you'll need to make sure that you can use it when you need to. In my class I make sure that the occasional problem requires this to make sure you don't get too locked into "nice" answers.

3. $x^2-8x+21=0$

Solution
Again, we'll need to use the quadratic formula for this one.

$$\begin{aligned} x &= \frac{8\pm\sqrt{64-4(1)(21)}}{2} \\ &= \frac{8\pm\sqrt{-20}}{2} \\ &= \frac{8\pm 2\sqrt{5}i}{2} \\ &= 4\pm\sqrt{5}i \end{aligned}$$

Complex numbers are a reality when solving equations, although we won't often see them in a Calculus class, if we see them at all.

Solving Equations, Part II

Solve each of the following equations for *y*.

1. $x = \dfrac{2y-5}{6-7y}$

Solution
Here all we need to do is get all the *y*'s on one side, factor a *y* out and then divide by the coefficient of the *y*.

$$\begin{aligned} x &= \frac{2y-5}{6-7y} \\ x\left(6-7y\right) &= 2y-5 \\ 6x-7xy &= 2y-5 \\ 6x+5 &= \left(7x+2\right)y \\ y &= \frac{6x+5}{7x+2} \end{aligned}$$

Solving equations for one of the variables in it is something that you'll be doing on occasion in a Calculus class so make sure that you can do it.

2. $3x^2\left(3-5y\right)+\sin x = 3xy+8$

Solution
This one solves the same way as the previous problem.

$$\begin{aligned} 3x^2\left(3-5y\right)+\sin x &= 3xy+8 \\ 9x^2-15x^2y+\sin x &= 3xy+8 \\ 9x^2+\sin x-8 &= \left(3x+15x^2\right)y \\ y &= \frac{9x^2+\sin x-8}{3x+15x^2} \end{aligned}$$

3. $2x^2+2y^2=5$

Solution
Same thing, just be careful with the last step.

$$\begin{aligned}2x^2+2y^2&=5\\2y^2&=5-2x^2\\y^2&=\frac{1}{2}\left(5-2x^2\right)\\y&=\pm\sqrt{\frac{5}{2}-x^2}\end{aligned}$$

Don't forget the "$\pm$" in the solution!

Solving Systems of Equations

1. Solve the following system of equations. Interpret the solution.

$$\begin{aligned}2x-y-2z&=-3\\x+3y+z&=-1\\5x-4y+3z&=10\end{aligned}$$

Solution

There are many possible ways to proceed in the solution process to this problem. All will give the same solution and all involve eliminating one of the variables and getting down to a system of two equations in two unknowns which you can then solve.

Method 1

The first solution method involves solving one of the three original equations for one of the variables. Substitute this into the other two equations. This will yield two equations in two unknowns that can be solve fairly quickly.

For this problem we'll solve the second equation for x to get.

$$x=-z-3y-1$$

Plugging this into the first and third equation gives the following system of two equations.

$$\begin{aligned}2\left(-z-3y-1\right)-y-2z&=-3\\5\left(-z-3y-1\right)-4y+3z&=10\end{aligned}$$

Or, upon simplification

$$\begin{aligned}-7y-4z&=-1\\-19y-2z&=15\end{aligned}$$

Multiply the second equation by -2 and add.

$$\begin{array}{r}-7y-4z=-1\\\underline{38y+4z=-30}\\31y=-31\end{array}$$

From this we see that $y=-1$. Plugging this into either of the above two equations yields $z=2$. Finally, plugging both of these answers into $x=-z-3y-1$ yields $x=0$.

Method 2
In the second method we add multiples of two equations together in such a way to eliminate one of the variables. We'll do it using two different sets of equations eliminating the same variable in both. This will give a system of two equations in two unknowns which we can solve.

So we'll start by noticing that if we multiply the second equation by -2 and add it to the first equation we get.

$$\begin{array}{r} 2x-y-2z=-3 \\ -2x-6y-2z=2 \\ \hline -7y-4z=-1 \end{array}$$

Next multiply the second equation by -5 and add it to the third equation. This gives

$$\begin{array}{r} -5x-15y-5z=5 \\ 5x-4y+3z=10 \\ \hline -19y-2z=15 \end{array}$$

This gives the following system of two equations.

$$\begin{array}{r} -7y-4z=-1 \\ -19y-2z=15 \end{array}$$

We can now solve this by multiplying the second by -2 and adding

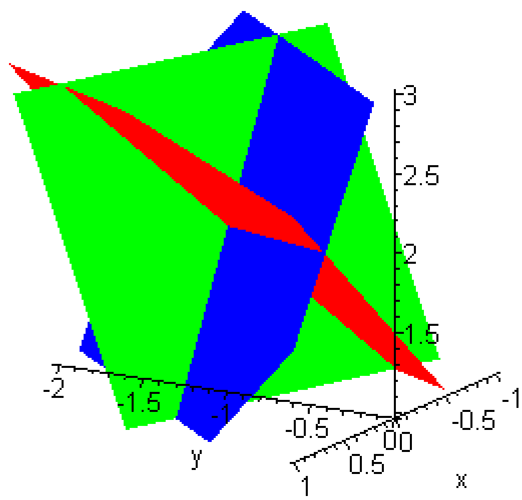
$$\begin{array}{r} -7y-4z=-1 \\ 38y+4z=-30 \\ \hline 31y=-31 \end{array}$$

From this we get that $y=-1$, the same as the first solution method. Plug this into either of the two equations involving only y and z and we'll get that $z=2$. Finally plug these into any of the original three equations and we'll get $x=0$.

You can use either of the two solution methods. In this case both methods involved the same basic level of work. In other cases on may be significantly easier than the other. You'll need to evaluate each system as you get it to determine which method will work the best.

Interpretation
Recall that one interpretation of the solution to a system of equations is that the solution(s) are the location(s) where the curves or surfaces (as in this case) intersect. So, the three equations in this system are the equations of planes in 3D space (you'll learn this in Calculus II if you don't already know this). So, from our solution we know that the three planes will intersect at the point (0,-1,2). Below is a graph of the three planes.

In this graph the red plane is the graph of $2x - y - 2z = -3$, the green plane is the graph of $x + 3y + z = 1$ and the blue plane is the graph of $5x - 4y + 3z = 10$. You can see from this figure that the three planes do appear to intersect at a single point. It is a somewhat hard to see what the exact coordinates of this point. However, if we could zoom in and get a better graph we would see that the coordinates of this point are $(0, -1, 2)$.

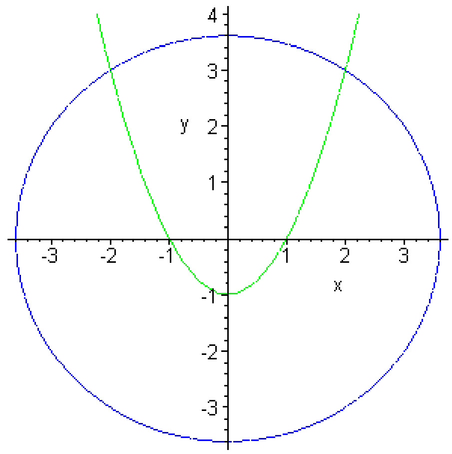


2. Determine where the following two curves intersect.

$$x^2 + y^2 = 13$$
$$y = x^2 - 1$$

Solution

In this case we're looking for where the circle and parabola intersect. Here's a quick graph to convince ourselves that they will in fact. This is not a bad idea to do with this kind of system. It is completely possible that the two curves don't cross and we would spend time trying to find a solution that doesn't exist! If intersection points do exist the graph will also tell us how many we can expect to get.



So, now that we know they cross let's proceed with finding the TWO intersection points. There are several ways to proceed at this point. One way would be to substitute the second equation into the first as follows then solve for *x*.

$$x^2 + \left(x^2 - 1\right)^2 = 13$$

While, this may be the way that first comes to mind it's probably more work than is required. Instead of doing it this way, let's rewrite the second equation as follows

$$x^2 = y + 1$$

Now, substitute this into the first equation.

$$y + 1 + y^2 = 13$$

$$y^2 + y - 12 = 0$$

$$(y+4)(y-3) = 0$$

This yields two values of *y*: $y = 3$ and $y = -4$. From the graph it's clear (I hope….) that no intersection points will occur for $y = -4$. However, let's suppose that we didn't have the graph and proceed without this knowledge. So, we will need to substitute each *y* value into one of the equations (I'll use $x^2 = y + 1$) and solve for *x*.

First, $y = 3$.

$$x^2 = 3 + 1 = 4$$

$$x = \pm 2$$

So, the two intersection points are (-2,3) and (2,3). Don't get used to these being "nice" answers, most of the time the solutions will be fractions and/or decimals.

Now, $y = -4$

$$x^2 = -4 + 1 = -3$$

$$x = \pm\sqrt{3}\, i$$

So, in this case we get complex solutions. This means that while $\left(\sqrt{3}\, i, -4\right)$ and $\left(-\sqrt{3}\, i, -4\right)$ are solutions to the system they do not correspond to intersection points.

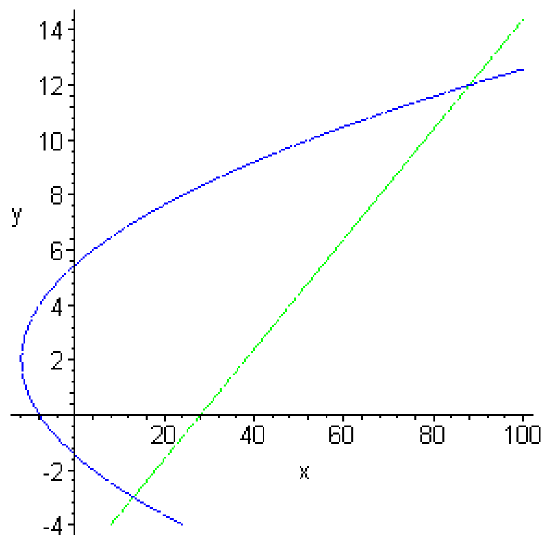
On occasion you will want the complex solutions and on occasion you won't want the complex solutions. You can usually tell from the problem statement or the type of problem that you are working if you need to include the complex solutions or not. In this case we were after where the two curves intersected which implies that we are after only the real solutions.

3. Graph the following two curves and determine where they intersect.

$$x = y^2 - 4y - 8$$
$$x = 5y + 28$$

Solution

Below is the graph of the two functions. If you don't remember how to graph function in the form $x = f(y)$ go back to the Graphing and Common Graphs section for quick refresher.



There are two intersection points for us to find. In this case, since both equations are of the form $x = f(y)$ we'll just set the two equations equal and solve for *y*.

$$y^2 - 4y - 8 = 5y + 28$$
$$y^2 - 9y - 36 = 0$$
$$(y+3)(y-12) = 0$$

The *y* coordinates of the two intersection points are then $y = -3$ and $y = 12$. Now, plug both of these into either of the original equations and solve for *x*. I'll use the line (second equation) since it seems a little easier.

First, $y = -3$.

$$x = 5(-3) + 28 = 13$$

Now, $y = 12$.

$$x = 5(12) + 28 = 88$$

So, the two intersection are (13,-3) and (88,12).

Solving Inequalities

Solve each of the following inequalities.

1. $x^2 - 10 > 3x$

Solution

To solve a polynomial inequality we get a zero on one side of the inequality, factor and then determine where the other side is zero.

$$x^2 - 10 > 3x \quad \Rightarrow \quad x^2 - 3x - 10 > 0 \quad \Rightarrow \quad (x-5)(x+2) > 0$$

So, once we move everything over to the left side and factor we can see that the left side will be zero at $x = 5$ and $x = -2$. These numbers are NOT solutions (since we only looking for values that will make the equation positive) but are useful to finding the actual solution.

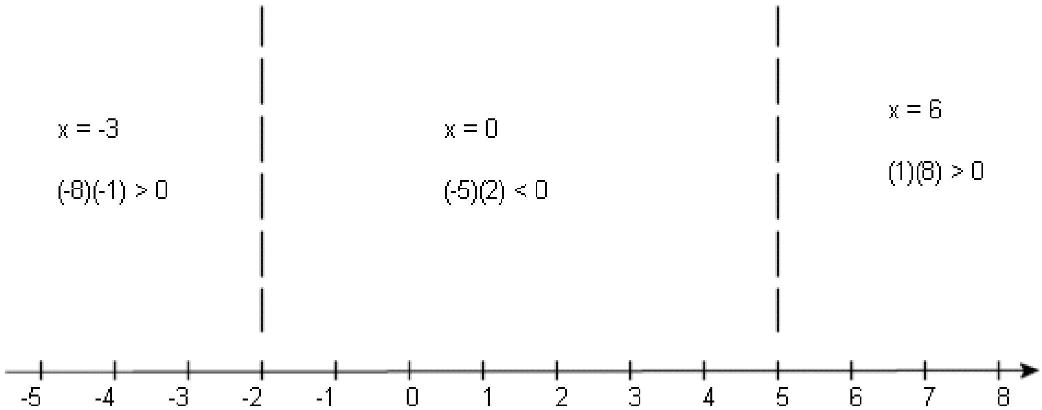
To find the solution to this inequality we need to recall that polynomials are nice smooth functions that have no breaks in them. This means that as we are moving across the number line (in any direction) if the value of the polynomial changes sign (say from positive to negative) then it MUST go through zero!

So, that means that these two numbers ($x = 5$ and $x = -2$) are the ONLY places where the polynomial can change sign. The number line is then divided into three regions. In each region if the inequality is satisfied by one point from that region then it is satisfied for ALL points in that region. If this wasn't true (*i.e* it was positive at one point in region and negative at another) then it must also be zero somewhere in that region, but that can't happen as we've already determined all the places where the polynomial can be zero! Likewise, if the inequality isn't satisfied for some point in that region that it isn't satisfied for ANY point in that region.

This means that all we need to do is pick a test point from each region (that are easy to work with, *i.e.* small integers if possible) and plug it into the inequality. If the test point satisfies the inequality then every point in that region does and if the test point doesn't satisfy the inequality then no point in that region does.

One final note here about this. I've got three versions of the inequality above. You can plug the test point into any of them, but it's usually easiest to plug the test points into the factored form of the inequality. So, if you trust your factoring capabilities that's the one to use. However, if you HAVE made a mistake in factoring, then you may end up with the incorrect solution if you use the factored form for testing. It's a trade-off. The factored form is, in many cases, easier to work with, but if you've made a mistake in factoring you may get the incorrect solution.

So, here's the number line and tests that I used for this problem.



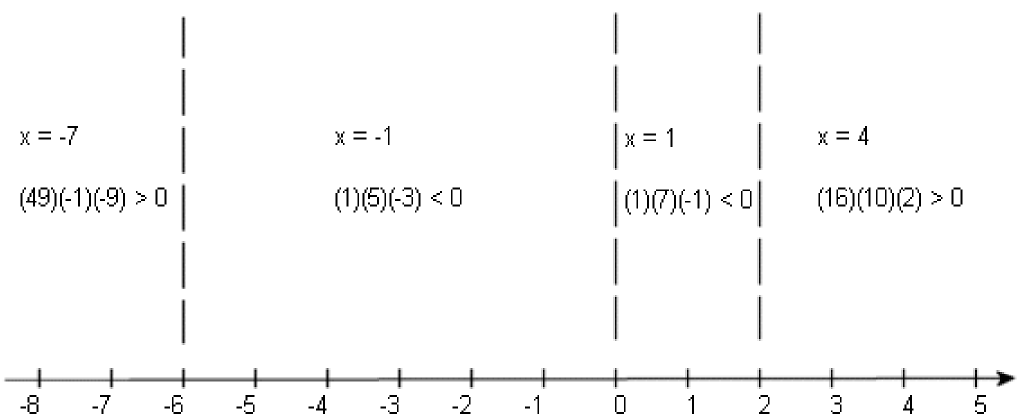
From this we see that the solution to this inequality is $-\infty < x < -2$ and $5 < x < \infty$. In interval notation this would be $(-\infty, -2)$ and $(5, \infty)$. You'll notice that the endpoints were not included in the solution for this. Pay attention to the original inequality when writing down the answer for these. Since the inequality was a strict inequality, we don't include the endpoints since these are the points that make both sides of the inequality equal!

2. $x^4 + 4x^3 - 12x^2 \le 0$

Solution
We'll do the same with this problem as the last problem.

$$x^4 + 4x^3 - 12x^2 \le 0 \quad \Rightarrow \quad x^2\left(x^2 + 4x - 12\right) \le 0 \quad \Rightarrow \quad x^2\left(x+6\right)\left(x-2\right) \le 0$$

In this case after factoring we can see that the left side will be zero at $x = -6$, $x = 0$ and $x = 2$.



From this number line the solution to the inequality is $-6 \le x \le 2$ or [-6,2]. Do not get locked into the idea that the intervals will alternate as solutions as they did in the first problem. Also, notice that in this case the inequality was less than OR EQUAL TO, so we did include the endpoints in our solution.

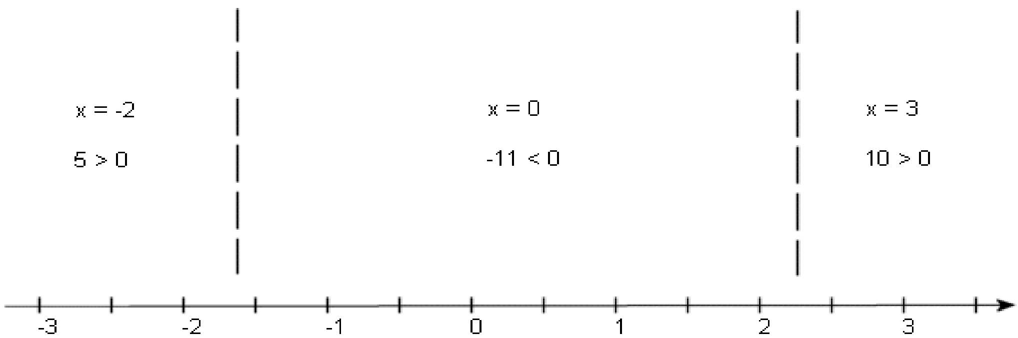
3. $3x^2 - 2x - 11 > 0$

Solution

This one is a little different, but not really more difficult. The quadratic doesn't factor and so we'll need to use the quadratic formula to solve for where it is zero. Doing this gives

$$x = \frac{1 \pm \sqrt{34}}{3}$$

Reducing to decimals this is $x = 2.27698$ and $x = -1.61032$. From this point on it's identical to the previous two problems. In the number line below the dashed lines are at the approximate values of the two numbers above and the inequalities show the value of the quadratic evaluated at the test points shown.



From the number line above we see that the solution is $\left(-\infty, \frac{1-\sqrt{34}}{3}\right)$ and $\left(\frac{1+\sqrt{34}}{3}, \infty\right)$.

4. $\dfrac{x-3}{x+2} \geq 0$

Solution

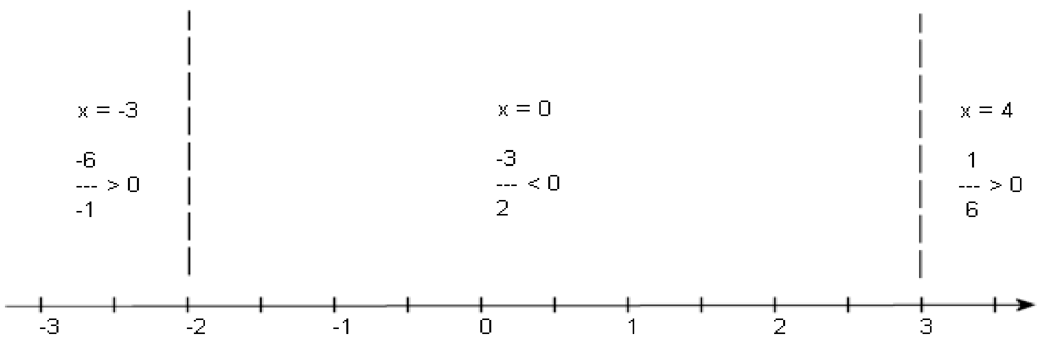
The process for solving inequalities that involve rational functions is nearly identical to solving inequalities that involve polynomials. Just like polynomial inequalities, rational inequalities can change sign where the rational expression is zero. However, they can also change sign at any point that produces a division by zero error in the rational expression. A good example of this is the rational expression $\frac{1}{x}$. Clearly, there is division by zero at $x = 0$ and to the right of $x = 0$ the expression is positive and to the left of $x = 0$ the expression is negative.

It's also important to note that a rational expression will only be zero for values of x that make the numerator zero.

So, what we need to do is first get a zero on one side of the inequality so we can use the above information. For this problem that has already been done. Now, determine where the numerator is zero (since the whole expression will be zero there) and where the denominator is zero (since we will get division by zero there).

At this point the process is identical to polynomial inequalities with one exception when we go to write down the answer. The points found above will divide the number line into regions in which the inequality will either always be true or always be false. So, pick test points from each region, test them in the inequality and get the solution from the results.

For this problem the numerator will be zero at $x = 3$ and the denominator will be zero at $x = -2$. The number line, along with the tests is shown below.



So, from this number line it looks like the two outer regions will satisfy the inequality. We need to be careful with the endpoints however. We will include $x = 3$ because this will make the rational expression zero and so will be part of the solution. On the other hand, $x = -2$ will give division by zero and so MUST be excluded from the solution since division by zero is never allowed.

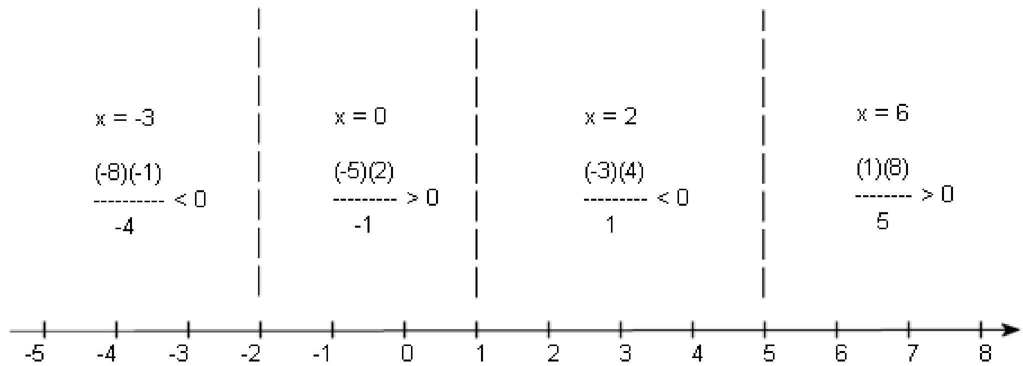
The solution to this inequality is $-\infty < x < -2$ and $3 \le x < \infty$ OR $(-\infty - 2)$ and $[3, \infty)$, depending on if you want inequality for the solution or intervals for the solution.

5. $\dfrac{x^2 - 3x - 10}{x - 1} < 0$

Solution
We already have a zero on one side of the inequality so first factor the numerator so we can get the points where the numerator will be zero.

$$\frac{(x-5)(x+2)}{x-1} < 0$$

So, the numerator will be zero at $x = -2$ and $x = 5$. The denominator will be zero at $x = 1$. The number line for this inequality is below.



In this case we won't include any endpoints in the solution since they either give division by zero or make the expression zero and we want strictly less than zero for this problem.

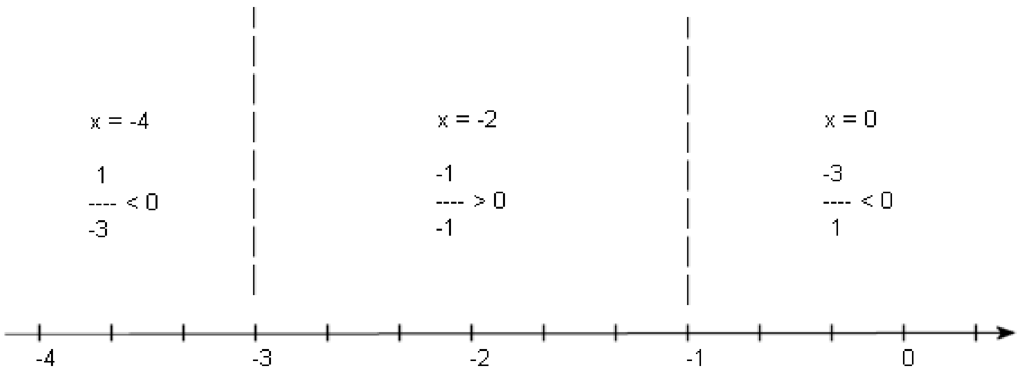
The solution is then $(-\infty,-2)$ and $(1,5)$.

6. $\dfrac{2x}{x+1} \geq 3$

Solution
We need to be a little careful with this one. In this case we need to get zero on one side of the inequality. This is easy enough to do. All we need to do is subtract 3 from both sides. This gives

$$\frac{2x}{x+1} - 3 \geq 0$$
$$\frac{2x-3(x+1)}{x+1} \geq 0$$
$$\frac{-x-3}{x+1} \geq 0$$

Notice that I also combined everything into a single rational expression. You will always want to do this. If you don't do this it can be difficult to determine where the rational expression is zero. So once we've gotten it into a single expression it's easy to see that the numerator will be zero at $x=-3$ and the denominator will be zero at $x=-1$. The number line for this problem is below.



The solution in this case is $[-3,-1)$. We don't include the -1 because this is where the solution is zero, but we do include the -3 because this makes the expression zero.

Absolute Value Equations and Inequalities

Solve each of the following.

1. $|3x+8|=2$

Solution
This uses the following fact

$$|p| = d \geq 0 \qquad \Rightarrow \qquad p = \pm d$$

The requirement that *d* be greater than or equal to zero is simply an acknowledgement that absolute value only returns number that are greater than or equal to zero. See Problem 3 below to see what happens when *d* is negative.

So the solution to this equation is

$$\begin{aligned} 3x+8&=2 \\ 3x&=-6 \\ x&=-2 \end{aligned} \qquad \text{OR} \qquad \begin{aligned} 3x+8&=-2 \\ 3x&=-10 \\ x&=-\frac{10}{3} \end{aligned}$$

So there were two solutions to this. That will almost always be the case. Also, do not get excited about the fact that these solutions are negative. This is not a problem. We can plug negative numbers into an absolute value equation (which is what we're doing with these answers), we just can't get negative numbers out of an absolute value (which we don't, we get 2 out of the absolute value in this case).

2. $|2x-4|=10$

Solution

This one works identically to the previous problem.

$$\begin{aligned} 2x-4&=10 & & & 2x-4&=-10 \\ 2x&=14 & &\text{OR} & 2x&=-6 \\ x&=7 & & & x&=-3 \end{aligned}$$

Do not make the following very common mistake in solve absolute value equations and inequalities.

$$\begin{aligned} \left|2x-4\right| \neq 2x+4 &= 10 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

Did you catch the mistake? In dropping the absolute value bars I just changed every "-" into a "+" and we know that doesn't work that way! By doing this we get a single answer and it's incorrect as well. Simply plug it into the original equation to convince yourself that it's incorrect.

$$\left|2(3)-4\right| = \left|6-4\right| = \left|2\right| = 2 \neq 10$$

When first learning to solve absolute value equations and inequalities people tend to just convert all minus signs to plus signs and solve. This is simply incorrect and will almost never get the correct answer. The way to solve absolute value equations is the way that I've shown here.

3. $\left|x+1\right| = -15$

Solution

This question is designed to make sure you understand absolute values. In this case we are after the values of x such that when we plug them into $\left|x+1\right|$ we will get -15. This is a problem however. Recall that absolute value ALWAYS returns a positive number! In other words, there is no way that we can get -15 out of this absolute value. Therefore, there are no solutions to this equation.

4. $\left|7x-10\right| \leq 4$

Solution

To solve absolute value inequalities with < or $\leq$ in them we use

$$\begin{aligned} \left|p\right| < d \qquad &\Rightarrow \qquad -d < p < d \\ \left|p\right| \leq d \qquad &\Rightarrow \qquad -d \leq p \leq d \end{aligned}$$

As with absolute value equations we will require that d be a number that is greater than or equal to zero.

The solution in this case is then

$$-4 \le 7x - 10 \le 4$$
$$6 \le 7x \le 14$$
$$\frac{6}{7} \le x \le 2$$

In solving these make sure that you remember to add the 10 to BOTH sides of the inequality and divide BOTH sides by the 7. One of the more common mistakes here is to just add or divide one side.

5. $|1-2x| < 7$

Solution
This one is identical to the previous problem with one small difference.

$$-7 < 1 - 2x < 7$$
$$-8 < -2x < 6$$
$$4 > x > -3$$

Don't forget that when multiplying or dividing an inequality by a negative number (-2 in this case) you've got to flip the direction of the inequality.

6. $|x-9| \le -1$

Solution
This problem is designed to show you how to deal with negative numbers on the other side of the inequality. So, we are looking for *x*'s which will give us a number (after taking the absolute value of course) that will be less than -1, but as with Problem 3 this just isn't possible since absolute value will always return a positive number or zero neither of which will ever be less than a negative number. So, there are no solutions to this inequality.

7. $|4x+5| > 3$

Solution
Absolute value inequalities involving > and $\ge$ are solved as follows.

$$|p| > d \qquad \Rightarrow \qquad p < -d \qquad \text{or} \qquad p > d$$
$$|p| \ge d \qquad \Rightarrow \qquad p \le -d \qquad \text{or} \qquad p \ge d$$

Note that you get two separate inequalities in the solution. That is the way that it must be. You can NOT put these together into a single inequality. Once I get the solution to this problem I'll show you why that is.

Here is the solution

$$\begin{aligned} 4x+5 &< -3 \\ 4x &< -8 \\ x &< -2 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 4x+5 &> 3 \\ 4x &> -2 \\ x &> -\frac{1}{2} \end{aligned}$$

So the solution to this inequality will be x's that are less than -2 or greater than $-\frac{1}{2}$.

Now, as I mentioned earlier you CAN NOT write the solution as the following double inequality.

$$-2 > x > -\frac{1}{2}$$

When you write a double inequality (as we have here) you are saying that x will be a number that will simultaneously satisfy both parts of the inequality. In other words, in writing this I'm saying that x is some number that is less than -2 and AT THE SAME TIME is greater than $-\frac{1}{2}$. I know of no number for which this is true. So, this is simply incorrect. Don't do it. This is however, a VERY common mistake that students make when solving this kinds of inequality.

8. $|4-11x| \geq 9$

Solution
Not much to this solution. Just be careful when you divide by the -11.

$$\begin{aligned} 4-11x &\leq -9 \\ -11x &\leq -13 \\ x &\geq \frac{13}{11} \end{aligned} \qquad \text{or} \qquad \begin{aligned} 4-11x &\geq 9 \\ -11x &\geq 5 \\ x &\leq -\frac{5}{11} \end{aligned}$$

9. $|10x+1| > -4$

Solution
This is another problem along the lines of Problems 3 and 6. However, the answer this time is VERY different. In this case we are looking for x's that when plugged in the absolute value we will get back an answer that is greater than -4, but since absolute value only return positive numbers or zero the result will ALWAYS be greater than any negative number. So, we can plug any x we would like into this absolute value and get a number greater than -4. So, the solution to this inequality is all real numbers.

Trigonometry

Trig Function Evaluation

One of the problems with most trig classes is that they tend to concentrate on right triangle trig and do everything in terms of degrees. Then you get to a calculus course where almost everything is done in radians and the unit circle is a very useful tool.

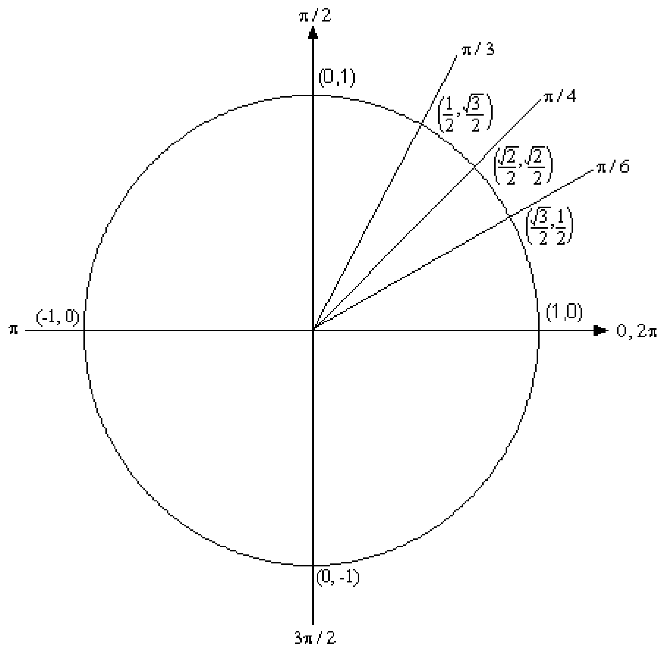
So first off let's look at the following table to relate degrees and radians.

Degree	0	30	45	60	90	180	270	360
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Know this table! There are, of course, many other angles in radians that we'll see during this class, but most will relate back to these few angles. So, if you can deal with these angles you will be able to deal with most of the others.

Be forewarned, everything in most calculus classes will be done in radians!

Now, let's look at the unit circle. Below is the unit circle with just the first quadrant filled in. The way the unit circle works is to draw a line from the center of the circle outwards corresponding to a given angle. Then look at the coordinates of the point where the line and the circle intersect. The first coordinate is the cosine of that angle and the second coordinate is the sine of that angle. There are a couple of *basic* angles that are commonly used. These are $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi, \frac{3\pi}{2},$ and 2π and are shown below along with the coordinates of the intersections. So, from the unit circle below we can see that $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.



Remember how the signs of angles work. If you rotate in a counter clockwise direction the angle is positive and if you rotate in a clockwise direction the angle is negative.

Recall as well that one complete revolution is 2π, so the positive *x*-axis can correspond to either an angle of 0 or 2π (or 4π, or 6π, or -2π, or -4π, *etc*. depending on the direction of rotation). Likewise, the angle $\frac{\pi}{6}$ (to pick an angle completely at random) can also be any of the following angles:

$\frac{\pi}{6}+2\pi=\frac{13\pi}{6}$ (start at $\frac{\pi}{6}$ then rotate once around counter clockwise)

$\frac{\pi}{6}+4\pi=\frac{25\pi}{6}$ (start at $\frac{\pi}{6}$ then rotate around twice counter clockwise)

$\frac{\pi}{6}-2\pi=-\frac{11\pi}{6}$ (start at $\frac{\pi}{6}$ then rotate once around clockwise)

$\frac{\pi}{6}-4\pi=-\frac{23\pi}{6}$ (start at $\frac{\pi}{6}$ then rotate around twice clockwise)

etc.

In fact $\frac{\pi}{6}$ can be any of the following angles $\frac{\pi}{6}+2\pi\, n$, $n=0,\pm 1,\pm 2,\pm 3,\ldots$ In this case n is the number of complete revolutions you make around the unit circle starting at $\frac{\pi}{6}$. Positive values of n correspond to counter clockwise rotations and negative values of n correspond to clockwise rotations.

So, why did I only put in the first quadrant? The answer is simple. If you know the first quadrant then you can get all the other quadrants from the first. You'll see this in the following examples.

Find the exact value of each of the following. In other words, don't use a calculator.

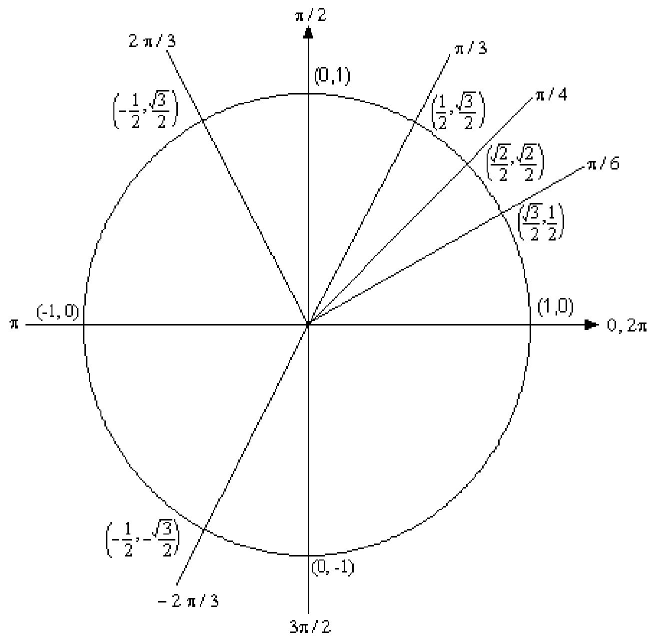
10. $\sin\left(\frac{2\pi}{3}\right)$ and $\sin\left(-\frac{2\pi}{3}\right)$

Solution

The first evaluation here uses the angle $\frac{2\pi}{3}$. Notice that $\frac{2\pi}{3}=\pi-\frac{\pi}{3}$. So $\frac{2\pi}{3}$ is found by rotating up $\frac{\pi}{3}$ from the negative x-axis. This means that the line for $\frac{2\pi}{3}$ will be a mirror image of the line for $\frac{\pi}{3}$ only in the second quadrant. The coordinates for $\frac{2\pi}{3}$ will be the coordinates for $\frac{\pi}{3}$ except the x coordinate will be negative.

Likewise for $-\frac{2\pi}{3}$ we can notice that $-\frac{2\pi}{3}=-\pi+\frac{\pi}{3}$, so this angle can be found by rotating down $\frac{\pi}{3}$ from the negative x-axis. This means that the line for $-\frac{2\pi}{3}$ will be a mirror image of the line for $\frac{\pi}{3}$ only in the third quadrant and the coordinates will be the same as the coordinates for $\frac{\pi}{3}$ except both will be negative.

Both of these angles along with their coordinates are shown on the following unit circle.



From this unit circle we can see that $\sin\left(\frac{2\pi}{3}\right)=\frac{\sqrt{3}}{2}$ and $\sin\left(-\frac{2\pi}{3}\right)=-\frac{\sqrt{3}}{2}$.

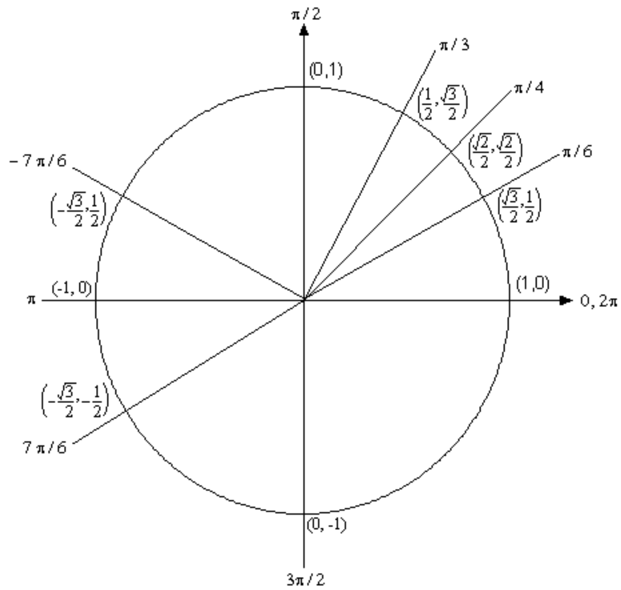
This leads to a nice fact about the sine function. The sine function is called an **odd** function and so for ANY angle we have

$$\sin(-\theta)=-\sin(\theta)$$

11. $\cos\left(\frac{7\pi}{6}\right)$ and $\cos\left(-\frac{7\pi}{6}\right)$

Solution

For this example notice that $\frac{7\pi}{6}=\pi+\frac{\pi}{6}$ so this means we would rotate down $\frac{\pi}{6}$ from the negative *x*-axis to get to this angle. Also $-\frac{7\pi}{6}=-\pi-\frac{\pi}{6}$ so this means we would rotate up $\frac{\pi}{6}$ from the negative *x*-axis to get to this angle. These are both shown on the following unit circle along with appropriate coordinates for the intersection points.



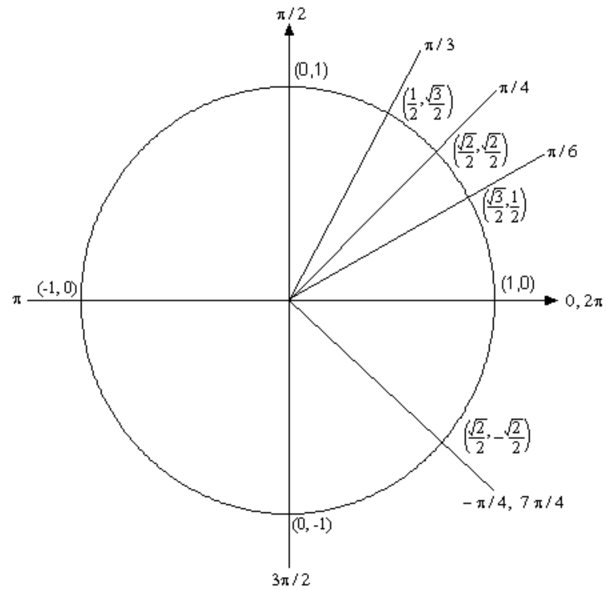
From this unit circle we can see that $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ and $\cos\left(-\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$. In this case the cosine function is called an **even** function and so for ANY angle we have

$$\cos(-\theta) = \cos(\theta).$$

12. $\tan\left(-\frac{\pi}{4}\right)$ and $\tan\left(\frac{7\pi}{4}\right)$

Solution

Here we should note that $\frac{7\pi}{4} = 2\pi - \frac{\pi}{4}$ so $\frac{7\pi}{4}$ and $-\frac{\pi}{4}$ are in fact the same angle! The unit circle for this angle is



Now, if we remember that $\tan\left(x\right)=\dfrac{\sin\left(x\right)}{\cos\left(x\right)}$ we can use the unit circle to find the values the tangent function. So,

$$\tan\left(\frac{7\pi}{4}\right)=\tan\left(-\frac{\pi}{4}\right)=\frac{\sin\left(-\pi/4\right)}{\cos\left(-\pi/4\right)}=\frac{-\sqrt{2}/2}{\sqrt{2}/2}=-1.$$

On a side note, notice that $\tan\left(\dfrac{\pi}{4}\right)=1$ and we se can see that the tangent function is also called an **odd** function and so for ANY angle we will have

$$\tan\left(-\theta\right)=-\tan\left(\theta\right).$$

13. $\sin\left(\dfrac{9\pi}{4}\right)$

Solution

For this problem let's notice that $\dfrac{9\pi}{4}=2\pi+\dfrac{\pi}{4}$. Now, recall that one complete revolution is 2π. So, this means that $\dfrac{9\pi}{4}$ and $\dfrac{\pi}{4}$ are at the same point on the unit circle. Therefore,

$$\sin\left(\frac{9\pi}{4}\right)=\sin\left(2\pi+\frac{\pi}{4}\right)=\sin\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$$

This leads us to a very nice fact about the sine function. The sine function is an example of a *periodic* function. Periodic function are functions that will repeat themselves over and over. The "distance" that you need to move to the right or left before the function starts repeating itself is called the **period** of the function.

In the case of sine the period is 2π. This means the sine function will repeat itself every 2π. This leads to a nice formula for the sine function.

$$\sin\left(x+2\pi n\right)=\sin\left(x\right) \qquad n=0,\pm 1,\ \pm 2,\ldots$$

Notice as well that since

$$\csc\left(x\right)=\frac{1}{\sin\left(x\right)}$$

we can say the same thing about cosecant.

$$\csc\left(x+2\pi n\right)=\csc\left(x\right) \qquad n=0,\pm 1,\ \pm 2,\ldots$$

Well, actually we should be careful here. We can say this provided $x\neq n\pi$ since sine will be zero at these points and so cosecant won't exist there!

14. $\sec\left(\dfrac{25\pi}{6}\right)$

Solution

Here we need to notice that $\dfrac{25\pi}{6}=4\pi+\dfrac{\pi}{6}$. In other words, we've started at $\dfrac{\pi}{6}$ and rotated around twice to end back up at the same point on the unit circle. This means that

$$\sec\left(\frac{25\pi}{6}\right)=\sec\left(4\pi+\frac{\pi}{6}\right)=\sec\left(\frac{\pi}{6}\right)$$

Now, let's also not get excited about the secant here. Just recall that

$$\sec\left(x\right)=\frac{1}{\cos\left(x\right)}$$

and so all we need to do here is evaluate a cosine! Therefore,

$$\sec\left(\frac{25\pi}{6}\right)=\sec\left(\frac{\pi}{6}\right)=\frac{1}{\cos\left(\frac{\pi}{6}\right)}=\frac{1}{\sqrt{3}/2}=\frac{2}{\sqrt{3}}$$

We should also note that cosine and secant are periodic functions with a period of 2π. So,

$$\begin{aligned}\cos\left(x+2\pi n\right)&=\cos\left(x\right)\\ \sec\left(x+2\pi n\right)&=\sec\left(x\right)\end{aligned} \qquad n=0,\pm 1,\ \pm 2,\ldots$$

15. $\tan\left(\dfrac{4\pi}{3}\right)$

Solution

To do this problem it will help to know that tangent (and hence cotangent) is also a periodic function, but unlike sine and cosine it has a period of π.

$$\begin{aligned}\tan(x+\pi n) &= \tan(x) \\ \cot(x+\pi n) &= \cot(x)\end{aligned} \qquad n = 0, \pm 1, \pm 2, \ldots$$

So, to do this problem let's note that $\frac{4\pi}{3} = \pi + \frac{\pi}{3}$. Therefore,

$$\tan\left(\frac{4\pi}{3}\right) = \tan\left(\pi + \frac{\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

Trig Evaluation Final Thoughts

As we saw in the previous examples if you know the first quadrant of the unit circle you can find the value of ANY trig function (not just sine and cosine) for ANY angle that can be related back to one of those shown in the first quadrant. This is a nice idea to remember as it means that you only need to memorize the first quadrant and how to get the angles in the remaining three quadrants!

In these problems I used only "basic" angles, but many of the ideas here can also be applied to angles other than these "basic" angles as we'll see in Solving Trig Equations.

Graphs of Trig Functions

There is not a whole lot to this section. It is here just to remind you of the graphs of the six trig functions as well as a couple of nice properties about trig functions.

Before jumping into the problems remember we saw in the Trig Function Evaluation section that trig functions are examples of *periodic* functions. This means that all we really need to do is graph the function for one periods length of values then repeat the graph.

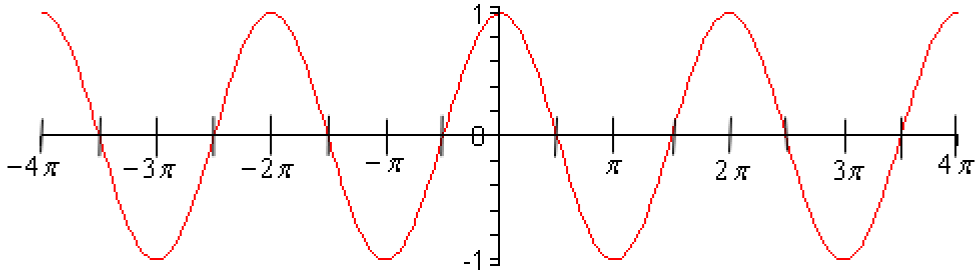
Graph the following function.

1. $y = \cos(x)$

Solution

There really isn't a whole lot to this one other than plotting a few points between 0 and 2π, then repeat. Remember cosine has a period of 2π (see Problem 6 in Trig Function Evaluation).

Here's the graph for $-4\pi \le x \le 4\pi$.



Notice that graph does repeat itself 4 times in this range of *x*'s as it should.

Let's also note here that we can put all values of *x* into cosine (which won't be the case for most of the trig functions) and let's also note that

$$-1 \le \cos(x) \le 1$$

It is important to notice that cosine will never be larger than 1 or smaller than -1. This will be useful on occasion in a calculus class.

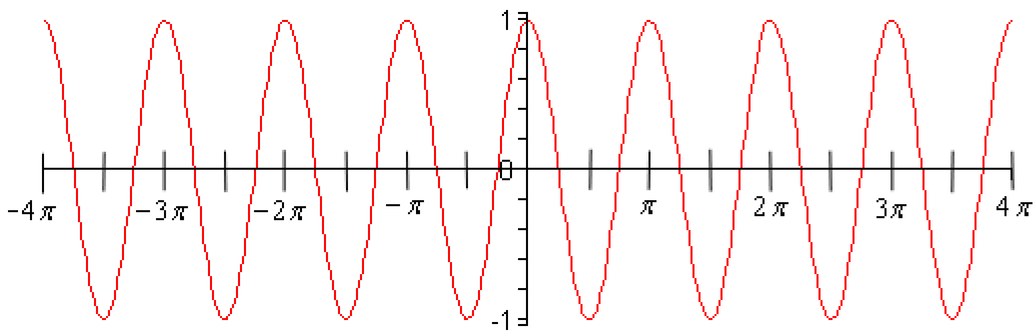
2. $y = \cos(2x)$

Solution

We need to be a little careful with this graph. $\cos(x)$ has a period of 2π, but we're not dealing with $\cos(x)$ here. We are dealing with $\cos(2x)$. In this case notice that if we plug in $x = \pi$ we will get

$$\cos(2(\pi)) = \cos(2\pi) = \cos(0) = 1$$

In this case the function starts to repeat itself after π instead of 2π! So, this function has a period of π. So, we can expect the graph to repeat itself 8 times in the range $-4\pi \le x \le 4\pi$. Here is that graph.



Sure enough, there are twice as many cycles in this graph.

In general we can get the period of $\cos(\omega x)$ using the following.

$$\text{Period} = \frac{2\pi}{\omega}$$

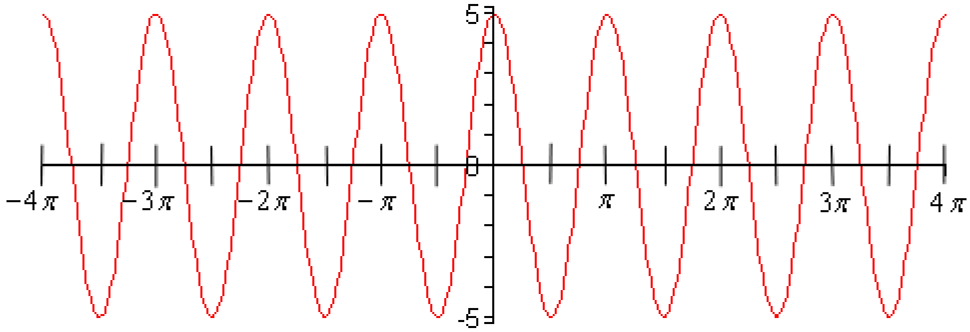
If $\omega > 1$ we can expect a period smaller than 2π and so the graph will oscillate faster. Likewise, if $\omega < 1$ we can expect a period larger than 2π and so the graph will oscillate slower.

Note that the period does not affect how large cosine will get. We still have

$$-1 \le \cos(2x) \le 1$$

3. $y = 5\cos(2x)$

Solution
In this case I added a 5 in front of the cosine. All that this will do is increase how big cosine will get. The number in front of the cosine or sine is called the **amplitude**. Here's the graph of this function.



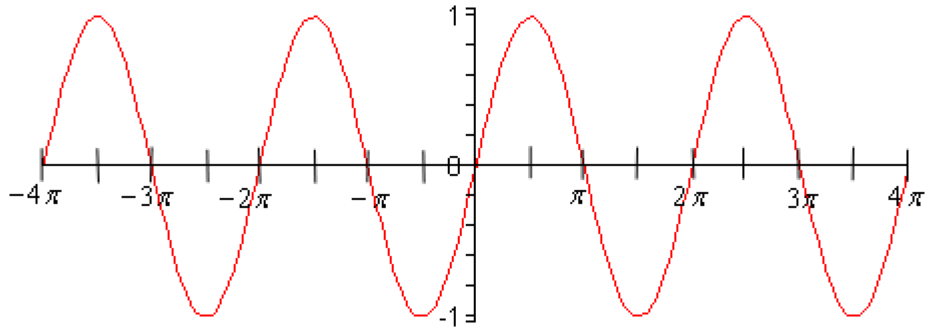
Note the scale on the *y*-axis for this problem and do not confuse it with the previous graph. The *y*-axis scales are different!

In general,

$$-R \le R\cos(\omega x) \le R$$

4. $y = \sin(x)$

Solution
As with the first problem in this section there really isn't a lot to do other than graph it. Here is the graph on the range $-4\pi \le x \le 4\pi$.



From this graph we can see that sine has the same range that cosine does. In general

$$-R \le R\sin\left(\omega x\right) \le R$$

As with cosine, sine itself will never be larger than 1 and never smaller than -1.

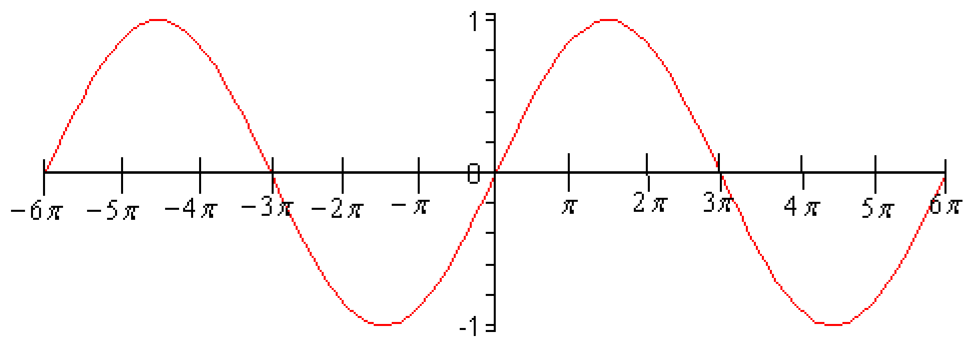
5. $y = \sin\left(\dfrac{x}{3}\right)$

Solution

So, in this case we don't have just an x inside the parenthesis. Just as in the case of cosine we can get the period of $\sin\left(\omega x\right)$ by using

$$\text{Period} = \frac{2\pi}{\omega} = \frac{2\pi}{{}^{1}\!/_{3}} = 6\pi$$

In this case the curve will repeat every 6π. So, for this graph I'll change the range to $-6\pi \le x \le 6\pi$ so we can get at least two traces of the curve showing. Here is the graph.



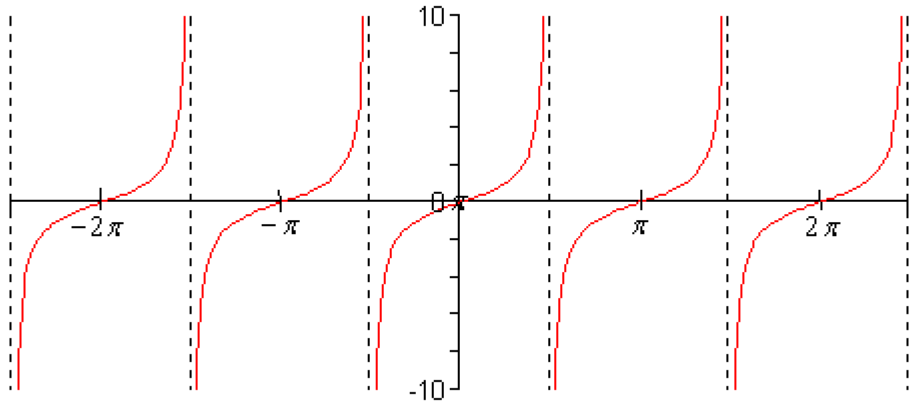
6. $y = \tan\left(x\right)$

Solution

In the case of tangent we have to be careful when plugging x's in since tangent doesn't exist wherever cosine is zero (remember that $\tan x = \dfrac{\sin x}{\cos x}$). Tangent will not exist at

$$x = \cdots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

and the graph will have asymptotes at these points. Here is the graph of tangent on the range $-\dfrac{5\pi}{2} < x < \dfrac{5\pi}{2}$.



Finally, a couple of quick properties about $R\tan(\omega x)$.

$$-\infty < R\tan(\omega x) < \infty$$

$$\text{Period} = \frac{\pi}{\omega}$$

For the period remember that $\tan(x)$ has a period of π unlike sine and cosine and that accounts for the absence of the 2 in the numerator that was there for sine and cosine.

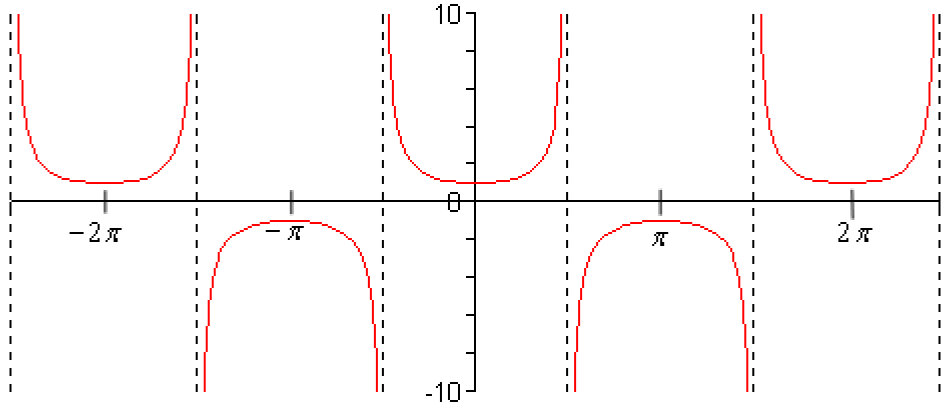
7. $y = \sec(x)$

Solution

As with tangent we will have to avoid x's for which cosine is zero (remember that $\sec x = \dfrac{1}{\cos x}$). Secant will not exist at

$$x = \cdots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

and the graph will have asymptotes at these points. Here is the graph of secant on the range $-\dfrac{5\pi}{2} < x < \dfrac{5\pi}{2}$.



Notice that the graph is always greater than 1 or less than -1. This should not be terribly surprising. Recall that $-1 \le \cos(x) \le 1$. So, 1 divided by something less than 1 will be greater than 1. Also, $1/\pm 1 = \pm 1$ and so we get the following ranges out of secant.

$$R\sec(\omega x) \ge R \qquad \text{and} \qquad R\sec(\omega x) \le -R$$

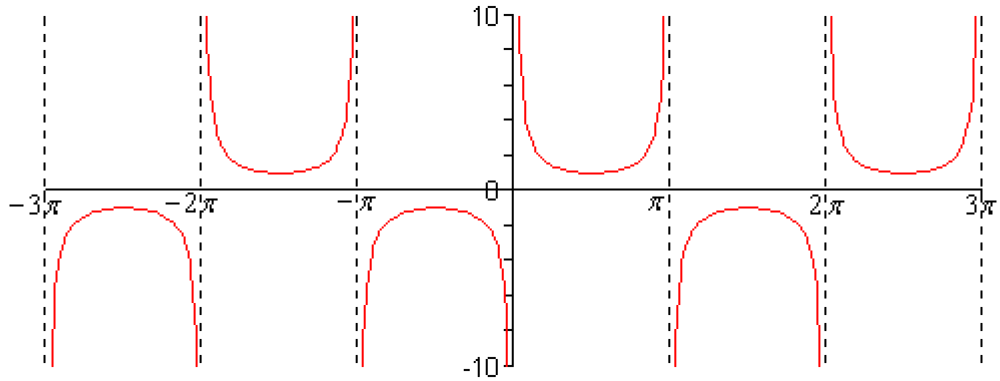
8. $y = \csc(x)$

Solution

For this graph we will have to avoid x's where sine is zero $\left(\csc x = \dfrac{1}{\sin x}\right)$. So, the graph of cosecant will not exist for

$$x = \cdots, -2\pi, -\pi, 0, \pi, 2\pi, \cdots$$

Here is the graph of cosecant.



Cosecant will have the same range as secant.

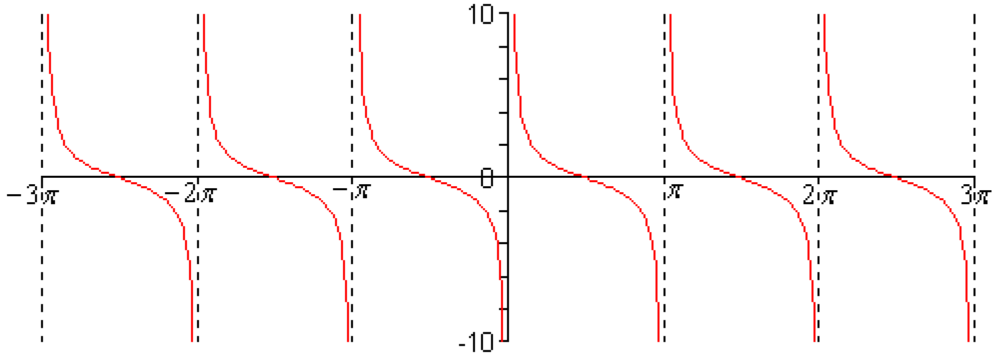
$$R\csc(\omega x) \ge R \qquad \text{and} \qquad R\csc(\omega x) \le -R$$

9. $y = \cot(x)$

Solution
Cotangent must avoid

$$x = \cdots, -2\pi, -\pi, 0, \pi, 2\pi, \cdots$$

since we will have division by zero at these points. Here is the graph.



Cotangent has the following range.

$$-\infty < R\cot(\omega x) < \infty$$

Trig Formulas

This is not a complete list of trig formulas. This is just a list of formulas that I've found to be the most useful in a Calculus class. For a complete listing of trig formulas you can download my Trig Cheat Sheet.

Complete the following formulas.

1. $\sin^2(\theta) + \cos^2(\theta) =$

Solution
$\sin^2(\theta) + \cos^2(\theta) = 1$

Note that this is true for ANY argument as long as it is the same in both the sine and the cosine. So, for example :

$$\sin^2(3x^4 - 5x^2 + 87) + \cos^2(3x^4 - 5x^2 + 87) = 1$$

2. $\tan^2(\theta) + 1 =$

Solution
$\tan^2(\theta) + 1 = \sec^2(\theta)$

If you know the formula from Problem 1 in this section you can get this one for free.

$$\begin{aligned}\sin^2(\theta)+\cos^2(\theta)&=1\\ \frac{\sin^2(\theta)}{\cos^2(\theta)}+\frac{\cos^2(\theta)}{\cos^2(\theta)}&=\frac{1}{\cos^2(\theta)}\\ \tan^2(\theta)+1&=\sec^2(\theta)\end{aligned}$$

Can you come up with a similar formula relating $\cot^2(\theta)$ and $\csc^2(\theta)$?

3. $\sin(2t)=$

Solution

$\sin(2t)=2\sin(t)\cos(t)$

This formula is often used in reverse so that a product of a sine and cosine (with the same argument of course) can be written as a single sine. For example,

$$\begin{aligned}\sin^3(3x^2)\cos^3(3x^2)&=\left(\sin(3x^2)\cos(3x^2)\right)^3\\ &=\left(\frac{1}{2}\sin\left(2\left(3x^2\right)\right)\right)^3\\ &=\frac{1}{8}\sin^3(6x^2)\end{aligned}$$

You will find that using this formula in reverse can significantly reduce the complexity of some of the problems that you'll face in a Calculus class.

4. $\cos(2x)=$ (Three possible formulas)

Solution

As noted there are three possible formulas to use here.

$$\begin{aligned}\cos(2x)&=\cos^2(x)-\sin^2(x)\\ \cos(2x)&=2\cos^2(x)-1\\ \cos(2x)&=1-2\sin^2(x)\end{aligned}$$

You can get the second formula by substituting $\sin^2(x)=1-\cos^2(x)$ (see Problem 1 from this section) into the first. Likewise, you can substitute $\cos^2(x)=1-\sin^2(x)$ into the first formula to get the third formula.

5. $\cos^2(x)=$ (In terms of cosine to the first power)

Solution

$$\cos^2(x) = \frac{1}{2}\left(1+\cos\left(2x\right)\right)$$

This is really the second formula from Problem 4 in this section rearranged and is VERY useful for eliminating even powers of cosines. For example,

$$\begin{aligned} 5\cos^2\left(3x\right) &= 5\left(\frac{1}{2}\left(1+\cos\left(2\left(3x\right)\right)\right)\right) \\ &= \frac{5}{2}\left(1+\cos\left(6x\right)\right) \end{aligned}$$

Note that you probably saw this formula written as

$$\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1}{2}\left(1+\cos\left(x\right)\right)}$$

in a trig class and called a half-angle formula.

6. $\sin^2(x) =$ (In terms of cosine to the first power)

Solution

$$\sin^2(x) = \frac{1}{2}\left(1-\cos\left(2x\right)\right)$$

As with the previous problem this is really the third formula from Problem 4 in this section rearranged and is very useful for eliminating even powers of sine. For example,

$$\begin{aligned} 4\sin^4\left(2t\right) &= 4\left(\sin^2\left(2t\right)\right)^2 \\ &= 4\left(\frac{1}{2}\left(1-\cos\left(4t\right)\right)\right)^2 \\ &= 4\left(\frac{1}{4}\right)\left(1-2\cos\left(4t\right)+\cos^2\left(4t\right)\right) \\ &= 1-2\cos\left(4t\right)+\frac{1}{2}\left(1+\cos\left(8t\right)\right) \\ &= \frac{3}{2}-2\cos\left(4t\right)+\frac{1}{2}\cos\left(8t\right) \end{aligned}$$

As shown in this example you may have to use both formulas and more than once if the power is larger than 2 and the answer will often have multiple cosines with different arguments.

Again, in a trig class this was probably called half-angle formula and written as,

$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1}{2}\left(1-\cos\left(x\right)\right)}$$

Solving Trig Equations

Solve the following trig equations. For those without intervals listed find ALL possible solutions. For those with intervals listed find only the solutions that fall in those intervals.

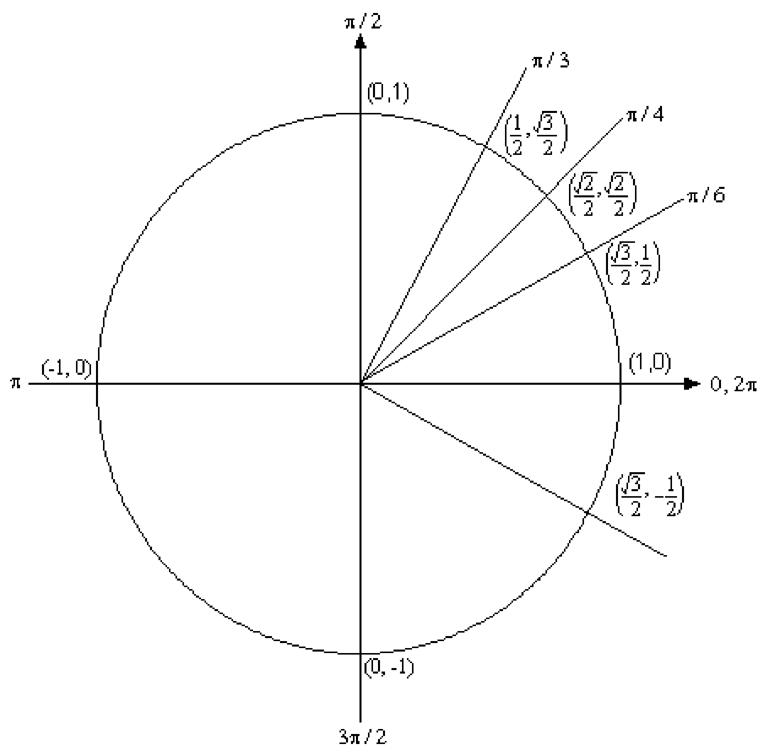
1. $2\cos(t) = \sqrt{3}$

Solution
There's not much to do with this one. Just divide both sides by 2 and then go to the unit circle.

$$2\cos(t) = \sqrt{3}$$
$$\cos(t) = \frac{\sqrt{3}}{2}$$

So, we are looking for all the values of *t* for which cosine will have the value of $\frac{\sqrt{3}}{2}$.

So, let's take a look at the following unit circle.



From quick inspection we can see that $t = \frac{\pi}{6}$ is a solution. However, as I have shown on the unit circle there is another angle which will also be a solution. We need to determine what this angle is. When we look for these angles we typically want *positive* angles that lie between 0 and 2π. This angle will not be the only possibility of course, but by convention we typically look for angles that meet these conditions.

To find this angle for this problem all we need to do is use a little geometry. The angle in the first quadrant makes an angle of $\frac{\pi}{6}$ with the positive *x*-axis, then so must the angle in the fourth quadrant. So we could use $-\frac{\pi}{6}$, but again, it's more common to use positive angles so, we'll use $t = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$.

We aren't done with this problem. As the discussion about finding the second angle has shown there are many ways to write any given angle on the unit circle. Sometimes it will be $-\frac{\pi}{6}$ that we want for the solution and sometimes we will want both (or neither) of the listed angles. Therefore, since there isn't anything in this problem (contrast this with the next problem) to tell us which is the correct solution we will need to list ALL possible solutions.

This is very easy to do. Go back to my introduction in the Trig Function Evaluation section and you'll see there that I used

$$\frac{\pi}{6} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

to represent all the possible angles that can end at the same location on the unit circle, *i.e.* angles that end at $\frac{\pi}{6}$. Remember that all this says is that we start at $\frac{\pi}{6}$ then rotate around in the counter-clockwise direction (*n* is positive) or clockwise direction (*n* is negative) for *n* complete rotations. The same thing can be done for the second solution.

So, all together the complete solution to this problem is

$$\frac{\pi}{6} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

$$\frac{11\pi}{6} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

As a final thought, notice that we can get $-\frac{\pi}{6}$ by using $n = -1$ in the second solution.

2. $2\cos(t) = \sqrt{3}$ on $[-2\pi, 2\pi]$

Solution

This problem is almost identical to the previous except now I want all the solutions that fall in the interval $[-2\pi, 2\pi]$. So we will start out with the list of all possible solutions from the previous problem.

$$\frac{\pi}{6} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$
$$\frac{11\pi}{6} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Then start picking values of *n* until we get all possible solutions in the interval.

First notice that since both the angles are positive *adding* on any multiples of 2π (*n* positive) will get us bigger than 2π and hence out of the interval. So, all positive values of *n* are immediately out. Let's take a look at the negatives values of *n*.

$$n = -1$$
$$\frac{\pi}{6} + 2\pi(-1) = -\frac{11\pi}{6} > -2\pi$$
$$\frac{11\pi}{6} + 2\pi(-1) = -\frac{\pi}{6} > -2\pi$$

These are both greater than -2π and so are solutions, but if we subtract another 2π off (*i.e* use $n = -2$) we will once again be outside of the interval.

So, the solutions are : $\frac{\pi}{6}, \frac{11\pi}{6}, -\frac{\pi}{6}, -\frac{11\pi}{6}$.

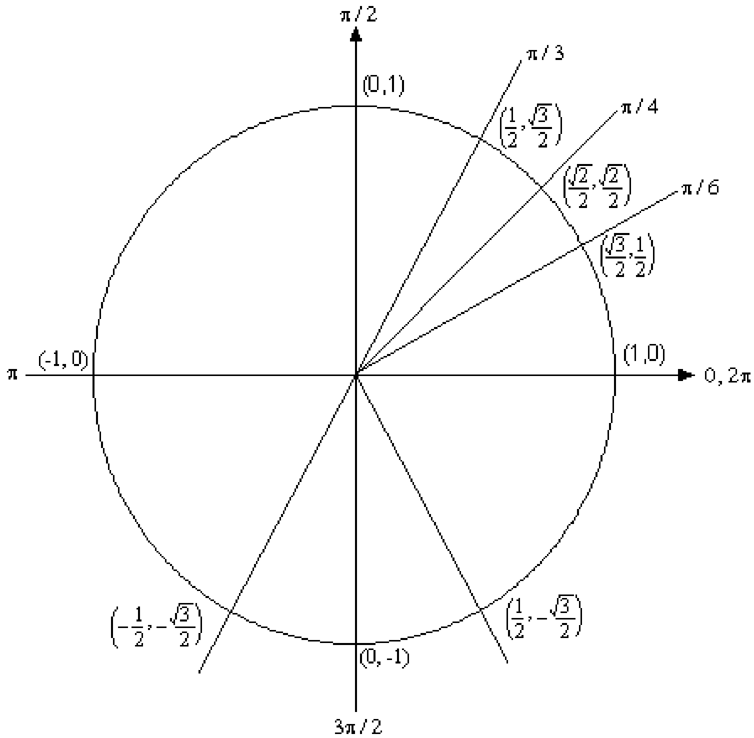
3. $2\sin(5x) = -\sqrt{3}$

Solution

This one is very similar to Problem 1, although there is a very important difference. We'll start this problem in exactly the same way as we did in Problem 1.

$$2\sin(5x) = -\sqrt{3}$$
$$\sin(5x) = \frac{-\sqrt{3}}{2}$$

So, we are looking for angles that will give $-\frac{\sqrt{3}}{2}$ out of the sine function. Let's again go to our trusty unit circle.



Now, there are no angles in the first quadrant for which sine has a value of $-\frac{\sqrt{3}}{2}$. However, there are two angles in the lower half of the unit circle for which sine will have a value of $-\frac{\sqrt{3}}{2}$. So, what are these angles? A quick way of doing this is to, for a second, ignore the "-" in the problem and solve $\sin\left(x\right)=\frac{\sqrt{3}}{2}$ in the first quadrant only. Doing this give a solution of $x=\frac{\pi}{3}$. Now, again using some geometry, this tells us that the angle in the third quadrant will be $\frac{\pi}{3}$ below the ***negative*** *x*-axis or $\pi+\frac{\pi}{3}=\frac{4\pi}{3}$. Likewise, the angle in the fourth quadrant will $\frac{\pi}{3}$ below the ***positive*** *x*-axis or $2\pi-\frac{\pi}{3}=\frac{5\pi}{3}$.

Now we come to the very important difference between this problem and Problem 1. The solution is **NOT**

$$x=\frac{4\pi}{3}+2\pi n, \quad n=0,\pm1,\pm2,\ldots$$
$$x=\frac{5\pi}{3}+2\pi n, \quad n=0,\pm1,\pm2,\ldots$$

This is not the set of solutions because we are NOT looking for values of x for which $\sin(x) = -\frac{\sqrt{3}}{2}$, but instead we are looking for values of x for which $\sin(5x) = -\frac{\sqrt{3}}{2}$. Note the difference in the arguments of the sine function! One is x and the other is $5x$. This makes all the difference in the world in finding the solution! Therefore, the set of solutions is

$$5x = \frac{4\pi}{3} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$5x = \frac{5\pi}{3} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Well, actually, that's not quite the solution. We are looking for values of x so divide everything by 5 to get.

$$x = \frac{4\pi}{15} + \frac{2\pi n}{5}, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$x = \frac{\pi}{3} + \frac{2\pi n}{5}, \quad n = 0, \pm 1, \pm 2, \ldots$$

Notice that I also divided the $2\pi n$ by 5 as well! This is important! If you don't do that you **WILL** miss solutions. For instance, take $n = 1$.

$$x = \frac{4\pi}{15} + \frac{2\pi}{5} = \frac{10\pi}{15} = \frac{2\pi}{3} \qquad \Rightarrow \qquad \sin\left(5\left(\frac{2\pi}{3}\right)\right) = \sin\left(\frac{10\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$x = \frac{\pi}{3} + \frac{2\pi}{5} = \frac{11\pi}{15} \qquad \Rightarrow \qquad \sin\left(5\left(\frac{11\pi}{15}\right)\right) = \sin\left(\frac{11\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

I'll leave it to you to verify my work showing they are solutions. However it makes the point. If you didn't divided the $2\pi n$ by 5 you would have missed these solutions!

4. $2\sin(5x+4) = -\sqrt{3}$

Solution

This problem is almost identical to the previous problem except this time we have an argument of $5x+4$ instead of $5x$. However, most of the problem is identical. In this case the solutions we get will be

$$5x + 4 = \frac{4\pi}{3} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$5x + 4 = \frac{5\pi}{3} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Notice the difference in the left hand sides between this solution and the corresponding solution in the previous problem.

Now we need to solve for x. We'll first subtract 4 from both sides then divide by 5 to get the following solution.

$$x = \frac{4\pi}{15} + \frac{2\pi n - 4}{5}, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$x = \frac{5\pi}{15} + \frac{2\pi n - 4}{5}, \quad n = 0, \pm 1, \pm 2, \ldots$$

It's somewhat messy, but it is the solution. Don't get excited when solutions get messy. They will on occasion and you need to get used to seeing them.

5. $2\sin(3x) = 1$ on $[-\pi, \pi]$

Solution

I'm going to leave most of the explanation that was in the previous three out of this one to see if you have caught on how to do these.

$$2\sin(3x) = 1$$
$$\sin(3x) = \frac{1}{2}$$

By examining a unit circle we see that

$$3x = \frac{\pi}{6} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$3x = \frac{5\pi}{6} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Or, upon dividing by 3,

$$x = \frac{\pi}{18} + \frac{2\pi n}{3}, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$x = \frac{5\pi}{18} + \frac{2\pi n}{3}, \quad n = 0, \pm 1, \pm 2, \ldots$$

Now, we are looking for solutions in the range $[-\pi, \pi]$. So, let's start trying some values of *n*.

$n = 0$ **:** $\qquad x = \frac{\pi}{18} \quad \& \quad x = \frac{5\pi}{18}$

$n = 1$ **:**

$$x = \frac{\pi}{18} + \frac{2\pi}{3} = \frac{13\pi}{18} < \pi \text{ so a solution}$$
$$x = \frac{5\pi}{18} + \frac{2\pi}{3} = \frac{17\pi}{18} < \pi \text{ so a solution}$$

$n = 2$ **:**

$$x = \frac{\pi}{18} + \frac{4\pi}{3} = \frac{25\pi}{18} > \pi \text{ so NOT a solution}$$
$$x = \frac{5\pi}{18} + \frac{4\pi}{3} = \frac{29\pi}{18} > \pi \text{ so NOT a solution}$$

Once, we've hit the limit in one direction there's no reason to continue on. In, other words if using $n=2$ gets values larger than π then so will all values of *n* larger than 2. Note as well that it is possible to have one of these be a solution and the other to not be a solution. It all depends on the interval being used.

Let's not forget the negative values of *n*.

$n=-1$ **:**

$$x=\frac{\pi}{18}-\frac{2\pi}{3}=-\frac{11\pi}{18}>-\pi \text{ so a solution}$$

$$x=\frac{5\pi}{18}-\frac{2\pi}{3}=-\frac{7\pi}{18}>-\pi \text{ so a solution}$$

$n=-2$ **:**

$$x=\frac{\pi}{18}-\frac{4\pi}{3}=-\frac{23\pi}{18}<-\pi \text{ so NOT a solution}$$

$$x=\frac{5\pi}{18}-\frac{4\pi}{3}=-\frac{19\pi}{18}<-\pi \text{ so NOT a solution}$$

Again, now that we've started getting less than $-\pi$ all other values of $n<-2$ will also give values that are less than $-\pi$.

So putting all this together gives the following six solutions.

$$x=-\frac{11\pi}{18},\frac{\pi}{18},\frac{13\pi}{18}$$

$$x=-\frac{7\pi}{18},\frac{5\pi}{18},\frac{17\pi}{18}$$

Finally, note again that if we hadn't divided the $2\pi n$ by 3 the only solutions we would have gotten would be $\frac{\pi}{18}$ and $\frac{5\pi}{18}$. We would have completely missed four of the solutions!

6. $\sin(4t)=1$ on $[0,2\pi]$

Solution

This one doesn't actually have a lot of work involved. We're looking for values of *t* for which $\sin(4t)=1$. This is one of the few trig equations for which there is only a single angle in all of $[0,2\pi]$ which will work. So our solutions are

$$4t=\frac{\pi}{2}+2\pi n, \quad n=0,\pm1,\pm2,\ldots$$

Or, by dividing by 4,

$$t=\frac{\pi}{8}+\frac{\pi n}{2}, \quad n=0,\pm1,\pm2,\ldots$$

Since we want the solutions on $[0, 2\pi]$ negative values of n aren't needed for this problem. So, plugging in values of n will give the following four solutions

$$t = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$$

7. $\cos(3x) = 2$

Solution
This is a trick question that is designed to remind you of certain properties about sine and cosine. Recall that $-1 \le \cos(\theta) \le 1$ and $-1 \le \sin(\theta) \le 1$. Therefore, since cosine will never be greater that 1 it definitely can't be 2. So **THERE ARE NO SOLUTIONS** to this equation!

8. $\sin(2x) = -\cos(2x)$

Solution
This problem is a little different from the previous ones. First, we need to do some rearranging and simplification.

$$\begin{aligned} \sin(2x) &= -\cos(2x) \\ \frac{\sin(2x)}{\cos(2x)} &= -1 \\ \tan(2x) &= -1 \end{aligned}$$

So, solving $\sin(2x) = -\cos(2x)$ is the same as solving $\tan(2x) = -1$. At some level we didn't need to do this for this problem as all we're looking for is angles in which sine and cosine have the same value, but opposite signs. However, for other problems this won't be the case and we'll want to convert to tangent.

Looking at our trusty unit circle it appears that the solutions will be,

$$2x = \frac{3\pi}{4} + 2\pi n, \quad n = 0, \pm1, \pm2, \ldots$$
$$2x = \frac{7\pi}{4} + 2\pi n, \quad n = 0, \pm1, \pm2, \ldots$$

Or, upon dividing by the 2 we get the solutions

$$x = \frac{3\pi}{8} + \pi n, \quad n = 0, \pm1, \pm2, \ldots$$
$$x = \frac{7\pi}{8} + \pi n, \quad n = 0, \pm1, \pm2, \ldots$$

No interval was given so we'll stop here.

9. $2\sin(\theta)\cos(\theta) = 1$

Solution

Again, we need to do a little work to get this equation into a form we can handle. The easiest way to do this one is to recall one of the trig formulas from the Trig Formulas section (in particular Problem 3).

$$2\sin(\theta)\cos(\theta)=1$$
$$\sin(2\theta)=1$$

At this point, proceed as we did in the previous problems..

$$2\theta=\frac{\pi}{2}+2\pi n,\quad n=0,\pm1,\pm2,\ldots$$

Or, by dividing by 2,

$$\theta=\frac{\pi}{4}+\pi n,\quad n=0,\pm1,\pm2,\ldots$$

Again, there is no interval so we stop here.

10. $\sin(w)\cos(w)+\cos(w)=0$

Solution

This problem is very different from the previous problems.

DO NOT DIVIDE BOTH SIDES BY A COSINE!!!!!!

If you divide both sides by a cosine you ***WILL*** lose solutions! The best way to deal with this one is to "factor" the equations as follows.

$$\sin(w)\cos(w)+\cos(w)=0$$
$$\cos(w)(\sin(w)+1)=0$$

So, solutions will be values of *w* for which

$$\cos(w)=0$$

or,

$$\sin(w)+1=0\qquad\Rightarrow\qquad\sin(w)=-1$$

In the first case we will have $\cos(w)=0$ at the following values.

$$w=\frac{\pi}{2}+2\pi n,\qquad n=0,\pm1,\pm2,\ldots$$
$$w=\frac{3\pi}{2}+2\pi n,\qquad n=0,\pm1,\pm2,\ldots$$

In the second case we will have $\sin(w)=-1$ at the following values.

$$w=\frac{3\pi}{2}+2\pi n,\quad n=0,\pm1,\pm2,\ldots$$

Note that in this case we got a repeat answer. Sometimes this will happen and sometimes it won't so don't expect this to always happen. So, all together we get the following solutions,

$$w = \frac{\pi}{2} + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots$$
$$w = \frac{3\pi}{2} + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots$$

As with the previous couple of problems this has no interval so we'll stop here. Notice as well that if we'd divided out a cosine we would have lost half the solutions.

11. $2\cos^2(3x) + 5\cos(3x) - 3 = 0$

Solution

This problem appears very difficult at first glance, but only the first step is different for the previous problems. First notice that

$$2t^2 + 5t - 3 = 0$$
$$(2t - 1)(t + 3) = 0$$

The solutions to this are $t = \frac{1}{2}$ and $t = -3$. So, why cover this? Well, if you think about it there is very little difference between this and the problem you are asked to do. First, we factor the equation

$$2\cos^2(3x) + 5\cos(3x) - 3 = 0$$
$$(2\cos(3x) - 1)(\cos(3x) + 3) = 0$$

The solutions to this are

$$\cos(3x) = \frac{1}{2} \qquad \text{and} \qquad \cos(3x) = -3$$

The solutions to the first are

$$3x = \frac{\pi}{3} + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots$$
$$3x = \frac{5\pi}{3} + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots$$

Or, upon dividing by 3,

$$x = \frac{\pi}{9} + \frac{2\pi n}{3}, \qquad n = 0, \pm 1, \pm 2, \ldots$$
$$x = \frac{5\pi}{9} + \frac{2\pi n}{3}, \qquad n = 0, \pm 1, \pm 2, \ldots$$

The second has no solutions because cosine can't be less that -1. Don't get used to this. Often both will yield solutions!

Therefore, the solutions to this are (again no interval so we're done at this point).

$$x = \frac{\pi}{9} + \frac{2\pi n}{3}, \qquad n = 0, \pm 1, \pm 2, \ldots$$
$$x = \frac{5\pi}{9} + \frac{2\pi n}{3}, \qquad n = 0, \pm 1, \pm 2, \ldots$$

12. $5\sin(2x) = 1$

Solution
This problem, in some ways, is VERY different from the previous problems and yet will work in essentially the same manner. To this point all the problems came down to a few "basic" angles that most people know and/or have used on a regular basis. This problem won't, but the solution process is pretty much the same. First, get the sine on one side by itself.

$$\sin(2x) = \frac{1}{5}$$

Now, at this point we know that we don't have one of the "basic" angles since those all pretty much come down to having 0, 1, $\frac{1}{2}$, $\frac{\sqrt{2}}{2}$ or $\frac{\sqrt{3}}{2}$ on the right side of the equal sign. So, in order to solve this we'll need to use our calculator. Every calculator is different, but most will have an inverse sine ($\sin^{-1}$), inverse cosine ($\cos^{-1}$) and inverse tangent ($\tan^{-1}$) button on them these days. If you aren't familiar with inverse trig functions see the next section in this review. Also, make sure that your calculator is set to do radians and not degrees for this problem.

It is also very important to understand the answer that your calculator will give. First, note that I said answer (*i.e.* a single answer) because that is all your calculator will ever give and we know from our work above that there are infinitely many answers. Next, when using your calculator to solve $\sin(x) = a$, *i.e.* using $\sin^{-1}(a)$, we will get the following ranges for *x*.

$$a \ge 0 \qquad \Rightarrow \qquad 0 \le x \le 1.570796327 = \frac{\pi}{2} \qquad (\text{Quad I})$$

$$a \le 0 \qquad \Rightarrow \qquad -\frac{\pi}{2} = -1.570796327 \le x \le 0 \qquad (\text{Quad IV})$$

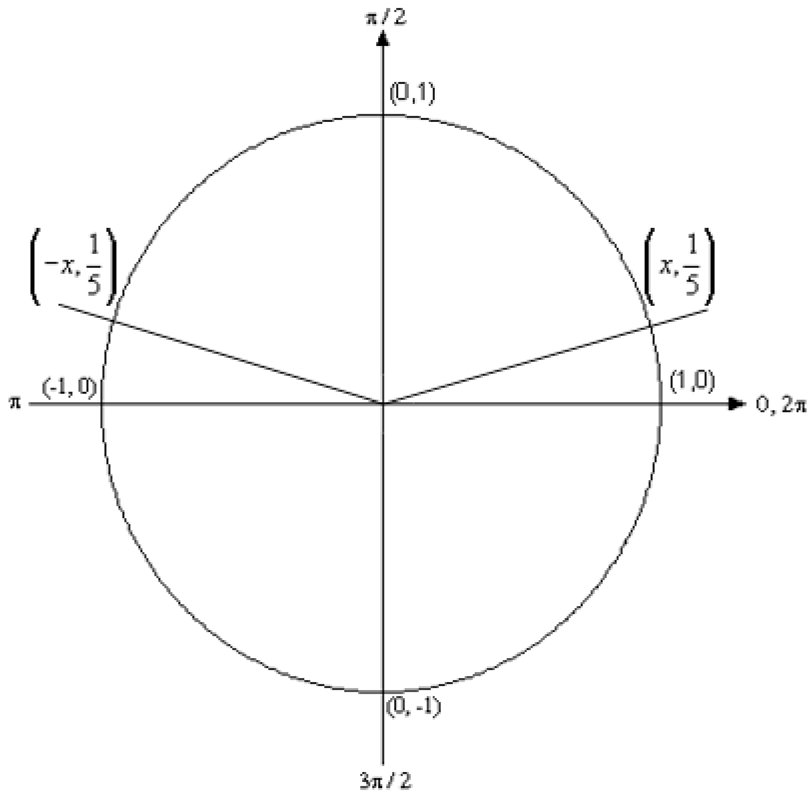
So, when using the inverse sine button on your calculator it will ONLY return answers in the first or fourth quadrant depending upon the sign of *a*.

Using our calculator in this problem yields,

$$2x = \sin^{-1}\left(\frac{1}{5}\right) = 0.2013579$$

Don't forget the 2 that is in the argument! We'll take care of that in a bit.

Now, we know from our work above that if there is a solution in the first quadrant to this equation then there will also be a solution in the second quadrant and that it will be at an angle of 0.2013579 above the *x*-axis as shown below.



I didn't put in the *x*, or cosine value, in the unit circle since it's not needed for the problem. I did however note that they will be the same value, except for the negative sign. The angle in the second quadrant will then be,

$$\pi - 0.2013579 = 2.9402348$$

So, let's put all this together.

$$2x = 0.2013579 + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots$$
$$2x = 2.9402348 + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots$$

Note that I added the $2\pi n$ onto our angles as well since we know that will be needed in order to get all the solutions. The final step is to then divide both sides by the 2 in order to get all possible solutions. Doing this gives,

$$x = 0.10067895 + \pi n, \qquad n = 0, \pm 1, \pm 2, \ldots$$
$$x = 1.4701174 + \pi n, \qquad n = 0, \pm 1, \pm 2, \ldots$$

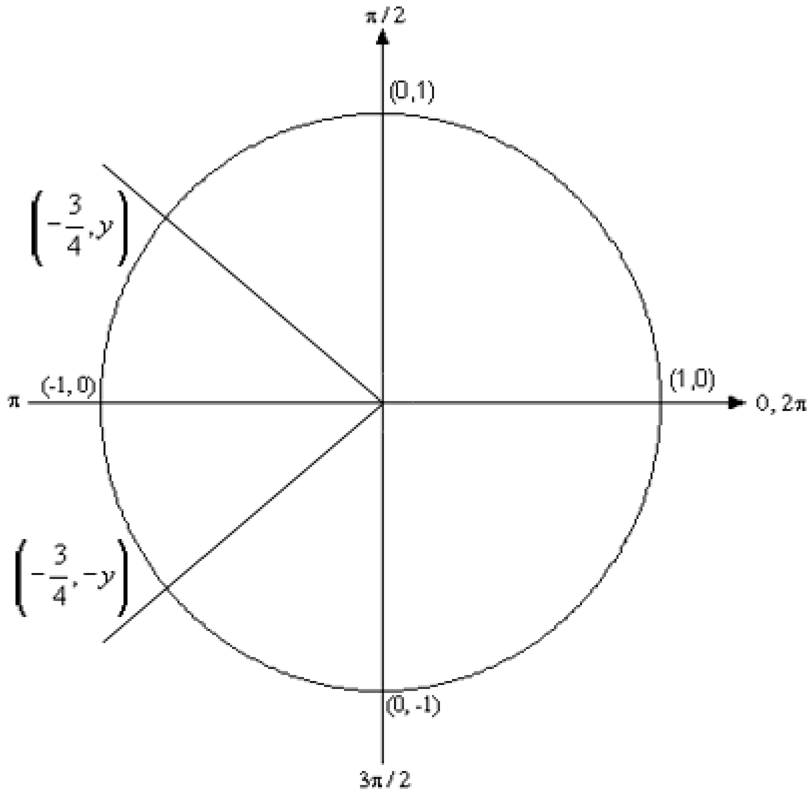
The answers won't be as "nice" as the answers in the previous problems but there they are. Note as well that if we'd been given an interval we could plug in values of *n* to determine the solutions that actually fall in the interval that we're interested in.

13. $4\cos\left(\frac{x}{5}\right) = -3$

Solution
This problem is again very similar to previous problems and yet has some differences. First get the cosine on one side by itself.

$$\cos\left(\frac{x}{5}\right) = -\frac{3}{4}$$

Now, let's take a quick look at a unit circle so we can see what angles we're after.



I didn't put the *y* values in since they aren't needed for this problem. Note however, that they will be the same except have opposite signs. Now, if this were a problem involving a "basic" angle we'd drop the "-" to determine the angle each of the lines above makes with the *x*-axis and then use that to find the actual angles. However, in this case since we're using a calculator we'll get the angle in the second quadrant for free so we may as well jump straight to that one.

However, prior to doing that let's acknowledge how the calculator will work when working with inverse cosines. If we're going to solve $\cos(x) = a$, using $\cos^{-1}(a)$, then our calculator will give one answer in one of the following ranges, depending upon the sign of *a*.

$$a \geq 0 \qquad \Rightarrow \qquad 0 \leq x \leq 1.570796 = \frac{\pi}{2} \qquad (\text{Quad I})$$

$$a \leq 0 \qquad \Rightarrow \qquad \pi = 3.141593 \leq x \leq 1.570796 = \frac{\pi}{2} \qquad (\text{Quad II})$$

So, using our calculator we get the following angle in the second quadrant.

$$\frac{x}{5} = \cos^{-1}\left(-\frac{3}{4}\right) = 2.418858$$

Now, we need to get the second angle that lies in the third quadrant. To find this angle note that the line in the second quadrant and the line in the third quadrant both make the same angle with the negative *x*-axis. Since we know what the angle in the second quadrant is we can find the angle that this line makes with the negative *x*-axis as follows,

$$\pi - 2.418858 = 0.722735$$

This means that the angle in the third quadrant is,

$$\pi + 0.722735 = 3.864328$$

Putting all this together gives,

$$\frac{x}{5} = 2.418858 + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\frac{x}{5} = 3.864328 + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Finally, we just need to multiply both sides by 5 to determine all possible solutions.

$$x = 12.09429 + 10\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$x = 19.32164 + 10\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

14. $10\sin(x-2) = -7$

Solution

We'll do this one much quicker than the previous two. First get the sine on one side by itself.

$$\sin(x-2) = -\frac{7}{10}$$

From a unit circle we can see that the two angles we'll be looking for are in the third and fourth quadrants. Our calculator will give us the angle that is in the fourth quadrant and this angle is,

$$x - 2 = \sin^{-1}\left(-\frac{7}{10}\right) = -0.775395$$

Note that in all the previous examples we generally wouldn't have used this answer because it is negative. There is nothing wrong with the answer, but as I mentioned several times in earlier problems we generally try to use positive angles between 0 and 2π. However, in this case since we are doing calculator work we won't worry about that fact that it's negative. If we wanted the positive angle we could always get it as,

$$2\pi - 0.775395 = 5.5077903$$

Now, the line corresponding to this solution makes an angle with the positive *x*-axis of 0.775395. The angle in the third quadrant will be 0.775395 radians below the negative *x*-axis and so is,

$$x - 2 = \pi + 0.775395 = 3.916988$$

Putting all this together gives,

$$\begin{aligned} x - 2 &= -0.775395 + 2\pi n, \qquad & n &= 0, \pm 1, \pm 2, \ldots \\ x - 2 &= 3.916988 + 2\pi n, \qquad & n &= 0, \pm 1, \pm 2, \ldots \end{aligned}$$

To get the final solution all we need to do is add 2 to both sides. All possible solutions are then,

$$\begin{aligned} x &= 1.224605 + 2\pi n, \quad & n &= 0, \pm 1, \pm 2, \ldots \\ x &= 5.916988 + 2\pi n, \quad & n &= 0, \pm 1, \pm 2, \ldots \end{aligned}$$

As the last three examples have shown, solving a trig equation that doesn't give any of the "basic" angles is not much different from those that do give "basic" angles. In fact, in some ways there are a little easier to do since our calculator will always give us one for free and all we need to do is find the second. The main idea here is to always remember that we need to be careful with our calculator and understand the results that it gives us.

Note as well that even for those problems that have "basic" angles as solutions we could have used a calculator as well. The only difference would have been that our answers would have been decimals instead of the exact answers we got.

Inverse Trig Functions

One of the more common notations for inverse trig functions can be very confusing. First, regardless of how you are used to dealing with exponentiation we tend to denote an inverse trig function with an "exponent" of "-1". In other words, the inverse cosine is denoted as $\cos^{-1}(x)$. It is important here to note that in this case the "-1" is NOT an exponent and so,

$$\cos^{-1}(x) \neq \frac{1}{\cos(x)}.$$

In inverse trig functions the "-1" looks like an exponent but it isn't, it is simply a notation that we use to denote the fact that we're dealing with an inverse trig function. It is a notation that we use in this case to denote inverse trig functions. If I had really wanted exponentiation to denote 1 over cosine I would use the following.

$$\left(\cos(x)\right)^{-1} = \frac{1}{\cos(x)}$$

There's another notation for inverse trig functions that avoids this ambiguity. It is the following.

$$\cos^{-1}(x) = \arccos(x)$$
$$\sin^{-1}(x) = \arcsin(x)$$
$$\tan^{-1}(x) = \arctan(x)$$

So, be careful with the notation for inverse trig functions!

There are, of course, similar inverse functions for the remaining three trig functions, but these are the main three that you'll see in a calculus class so I'm going to concentrate on them.

To evaluate inverse trig functions remember that the following statements are equivalent.

$$\theta = \cos^{-1}(x) \qquad \Leftrightarrow \qquad x = \cos(\theta)$$
$$\theta = \sin^{-1}(x) \qquad \Leftrightarrow \qquad x = \sin(\theta)$$
$$\theta = \tan^{-1}(x) \qquad \Leftrightarrow \qquad x = \tan(\theta)$$

In other words, when we evaluate an inverse trig function we are asking what angle, θ, did we plug into the trig function (regular, not inverse!) to get x.

So, let's do some problems to see how these work. Evaluate each of the following.

1. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Solution

In Problem 1 of the [Solving Trig Equations] section we solved the following equation.

$$\cos(t) = \frac{\sqrt{3}}{2}$$

In other words, we asked what angles, t, do we need to plug into cosine to get $\frac{\sqrt{3}}{2}$?

This is essentially what we are asking here when we are asked to compute the inverse trig function.

$$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

There is one very large difference however. In Problem 1 we were solving an equation which yielded an infinite number of solutions. These were,

$$\frac{\pi}{6} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$
$$\frac{11\pi}{6} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

In the case of inverse trig functions we are after a single value. We don't want to have to guess at which one of the infinite possible answers we want. So, to make sure we get a single value out of the inverse trig cosine function we use the following restrictions on inverse cosine.

$$\theta = \cos^{-1}(x) \qquad -1 \le x \le 1 \qquad \text{and} \qquad 0 \le \theta \le \pi$$

The restriction on the θ guarantees that we will only get a single value angle and since we can't get values of x out of cosine that are larger than 1 or smaller than -1 we also can't plug these values into an inverse trig function.

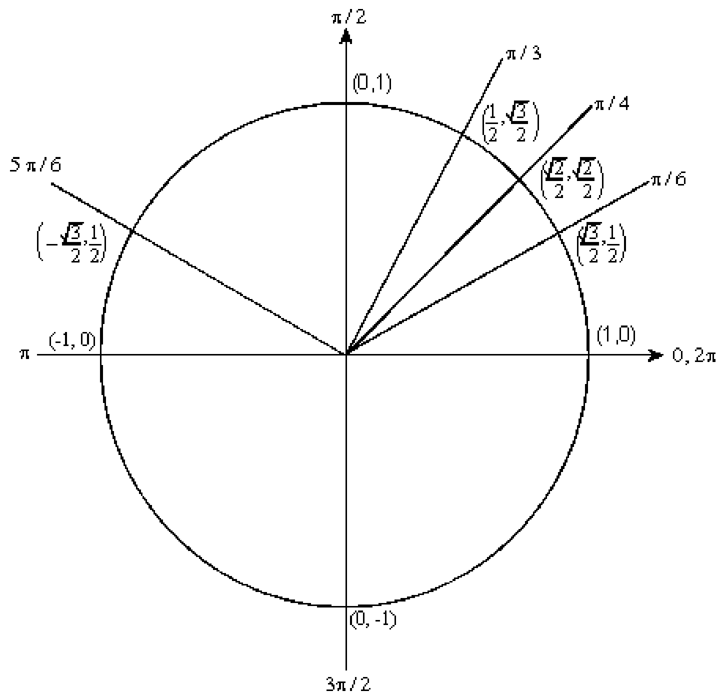
So, using these restrictions on the solution to Problem 1 we can see that the answer in this case is

$$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

2. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

Solution

In general we don't need to actually solve an equation to determine the value of an inverse trig function. All we need to do is look at a unit circle. So in this case we're after an angle between 0 and π for which cosine will take on the value $-\frac{\sqrt{3}}{2}$. So, check out the following unit circle



From this we can see that

$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\frac{5\pi}{6}$$

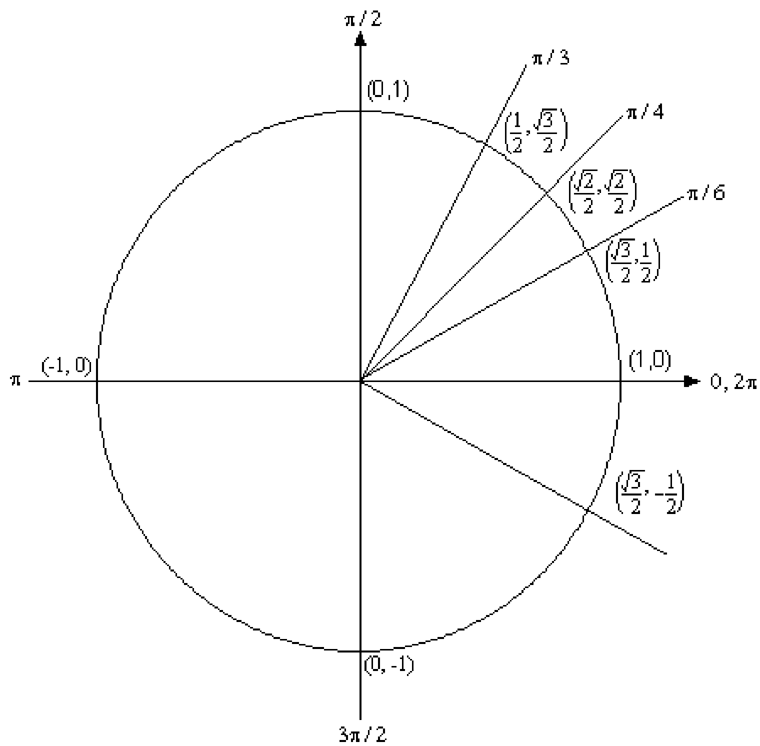
3. $\sin^{-1}\left(-\frac{1}{2}\right)$

Solution

The restrictions that we put on θ for the inverse cosine function will not work for the inverse sine function. Just look at the unit circle above and you will see that between 0 and π there are in fact two angles for which sine would be $\frac{1}{2}$ and this is not what we want. As with the inverse cosine function we only want a single value. Therefore, for the inverse sine function we use the following restrictions.

$$\theta=\sin^{-1}\left(x\right) \qquad -1\le x\le 1 \qquad \text{and} \qquad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$$

By checking out the unit circle



we see

$$\sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$$

4. $\tan^{-1}(1)$

Solution
The restriction for inverse tangent is

$$\theta=\tan^{-1}(x) \qquad\qquad -\frac{\pi}{2}<\theta<\frac{\pi}{2}$$

Notice that there is no restriction on *x* this time. This is because $\tan(\theta)$ can take any value from negative infinity to positive infinity. If this is true then we can also plug any value into the inverse tangent function. Also note that we don't include the two endpoints on the restriction on θ. Tangent is not defined at these two points, so we can't plug them into the inverse tangent function.

In this problem we're looking for the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ for which $\tan(\theta)=1$, or $\sin(\theta)=\cos(\theta)$. This can only occur at $\theta=\frac{\pi}{4}$ so,

$$\tan^{-1}(1)=\frac{\pi}{4}$$

5. $\cos\left(\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$

Solution
Recalling the answer to Problem 1 in this section the solution to this problem is much easier than it look's like on the surface.

$$\cos\left(\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)=\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$$

This problem leads to a couple of nice facts about inverse cosine

$$\cos\left(\cos^{-1}(x)\right)=x \qquad \text{AND} \qquad \cos^{-1}\left(\cos(\theta)\right)=\theta$$

6. $\sin^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right)$

Solution
This problem is also not too difficult (hopefully…).

$$\sin^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right)=\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)=\frac{\pi}{4}$$

As with inverse cosine we also have the following facts about inverse sine.

$$\sin\left(\sin^{-1}(x)\right)=x \qquad \text{AND} \qquad \sin^{-1}\left(\sin(\theta)\right)=\theta$$

7. $\tan\left(\tan^{-1}(-4)\right)$.

Solution
Just as inverse cosine and inverse sine had a couple of nice facts about them so does inverse tangent. Here is the fact

$$\tan\left(\tan^{-1}(x)\right)=x \qquad \text{AND} \qquad \tan^{-1}\left(\tan(\theta)\right)=\theta$$

Using this fact makes this a very easy problem as I couldn't do $\tan^{-1}(4)$ by hand! A calculator could easily do it but I couldn't get an exact answer from a unit circle.

$$\tan\left(\tan^{-1}(-4)\right)=-4$$

Exponentials / Logarithms

Basic Exponential Functions

First, let's recall that for $b > 0$ and $b \neq 1$ an exponential function is any function that is in the form

$$f(x) = b^x$$

We require $b \neq 1$ to avoid the following situation,

$$f(x) = 1^x = 1$$

So, if we allowed $b = 1$ we would just get the constant function, 1.

We require $b > 0$ to avoid the following situation,

$$f(x) = (-4)^x \qquad \Rightarrow \qquad f\left(\frac{1}{2}\right) = (-4)^{\frac{1}{2}} = \sqrt{-4}$$

By requiring $b > 0$ we don't have to worry about the possibility of square roots of negative numbers.

1. Evaluate $f(x) = 4^x$, $g(x) = \left(\frac{1}{4}\right)^x$ and $h(x) = 4^{-x}$ at $x = -2, -1, 0, 1, 2$.

Solution

The point here is mostly to make sure you can evaluate these kinds of functions. So, here's a quick table with the answers.

	$x=-2$	$x=-1$	$x=0$	$x=1$	$x=2$
$f(x)$	$f(-2)=\frac{1}{16}$	$f(-1)=\frac{1}{4}$	$f(0)=1$	$f(1)=4$	$f(2)=16$
$g(x)$	$g(-2)=16$	$g(-1)=4$	$g(0)=1$	$g(1)=\frac{1}{4}$	$g(2)=\frac{1}{16}$
$h(x)$	$h(-2)=16$	$h(-1)=4$	$h(0)=1$	$h(1)=\frac{1}{4}$	$h(2)=\frac{1}{16}$

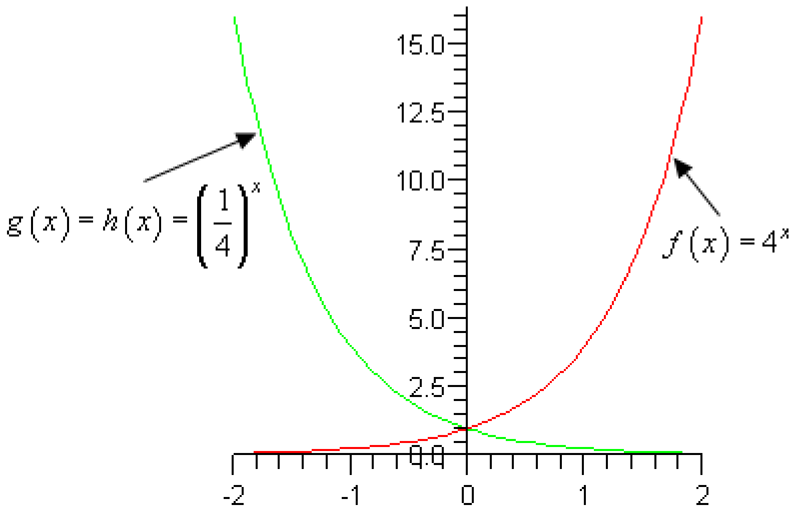
Notice that the last two rows give exactly the same answer. If you think about it that should make sense because,

$$g(x) = \left(\frac{1}{4}\right)^x = \frac{1^x}{4^x} = \frac{1}{4^x} = 4^{-x} = h(x)$$

2. Sketch the graph of $f(x) = 4^x$, $g(x) = \left(\frac{1}{4}\right)^x$ and $h(x) = 4^{-x}$ on the same axis system.

Solution

Note that we only really need to graph $f(x)$ and $g(x)$ since we showed in the previous Problem that $g(x) = h(x)$. Note as well that there really isn't too much to do here. We found a set of values in Problem 1 so all we need to do is plot the points and then sketch the graph. Here is the sketch,



3. List as some basic properties for $f(x) = b^x$.

Solution
Most of these properties can be seen in the sketch in the previous Problem.

(a) $f(x) = b^x > 0$ for every *x*. This is a direct consequence of the requirement that $b > 0$.

(b) For any *b* we have $f(0) = b^0 = 1$.

(c) If $b > 1$ ($f(x) = 4^x$ above, for example) we see that $f(x) = b^x$ is an increasing function and that,

$$f(x) \to \infty \text{ as } x \to \infty \qquad \text{and} \qquad f(x) \to 0 \text{ as } x \to -\infty$$

(d) If $0 < b < 1$ ($g(x) = \left(\frac{1}{4}\right)^x$ above, for example) we see that $f(x) = b^x$ is an decreasing function and that,

$$f(x) \to 0 \text{ as } x \to \infty \qquad \text{and} \qquad f(x) \to \infty \text{ as } x \to -\infty$$

Note that the last two properties are very important properties in many Calculus topics and so you should always remember them!

4. Evaluate $f(x) = \mathbf{e}^x$, $g(x) = \mathbf{e}^{-x}$ and $h(x) = 5\mathbf{e}^{1-3x}$ at $x = -2, -1, 0, 1, 2$.

Solution

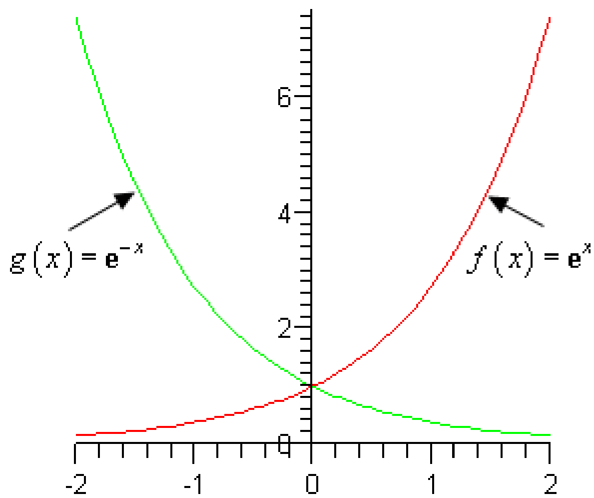
Again, the point of this problem is to make sure you can evaluate these kinds of functions. Recall that in these problems **e** is not a variable it is a number! In fact, $\mathbf{e} = 2.718281828\ldots$

When computing $h(x)$ make sure that you do the exponentiation BEFORE multiplying by 5.

	$x=-2$	$x=-1$	$x=0$	$x=1$	$x=2$
$f(x)$	0.135335	0.367879	1	2.718282	7.389056
$g(x)$	7.389056	2.718282	1	0.367879	0.135335
$h(x)$	5483.166	272.9908	13.59141	0.676676	0.033690

5. Sketch the graph of $f(x) = \mathbf{e}^x$ and $g(x) = \mathbf{e}^{-x}$.

Solution
As with the other "sketching" problem there isn't much to do here other than use the numbers we found in the previous example to make the sketch. Here it is,



Note that from these graphs we can see the following important properties about $f(x) = \mathbf{e}^x$ and $g(x) = \mathbf{e}^{-x}$.

$\mathbf{e}^x \to \infty$ as $x \to \infty$ and $\mathbf{e}^x \to 0$ as $x \to -\infty$

$\mathbf{e}^{-x} \to 0$ as $x \to \infty$ and $\mathbf{e}^{-x} \to \infty$ as $x \to -\infty$

These properties show up with some regularity in a Calculus course and so should be remembered.

Basic Logarithmic Functions

1. Without a calculator give the exact value of each of the following logarithms.

(a) $\log_2 16$ (b) $\log_4 16$ (c) $\log_5 625$

(d) $\log_9 \frac{1}{531441}$ (e) $\log_{\frac{1}{6}} 36$ (f) $\log_{\frac{3}{2}} \frac{27}{8}$

Solution

To do these without a calculator you need to remember the following.

$y = \log_b x$ is equivalent to $x = b^y$

Where, *b*, is called the base is any number such that $b > 0$ and $b \neq 1$. The first is usually called **logarithmic form** and the second is usually called **exponential form**. The logarithmic form is read "*y* equals log base *b* of *x*".

So, if you convert the logarithms to exponential form it's usually fairly easy to compute these kinds of logarithms.

(i) $\log_2 16 = 4$ because $2^4 = 16$

(j) $\log_4 16 = 2$ because $4^2 = 16$

Note the difference between **(a)** and **(b)**! The base, *b*, that you use on the logarithm is VERY important! A different base will, in almost every case, yield a different answer. You should always pay attention to the base!

(k) $\log_5 625 = 4$ because $5^4 = 625$

(l) $\log_9 \frac{1}{531441} = -6$ because $9^{-6} = \frac{1}{9^6} = \frac{1}{531441}$

(m) $\log_{\frac{1}{6}} 36 = -2$ because $\left(\frac{1}{6}\right)^{-2} = 6^2 = 36$

(n) $\log_{\frac{3}{2}} \frac{27}{8} = 3$ because $\left(\frac{3}{2}\right)^3 = \frac{27}{8}$

2. Without a calculator give the exact value of each of the following logarithms.

(a) $\ln \sqrt[3]{\mathbf{e}}$ (b) $\log 1000$ (c) $\log_{16} 16$

(d) $\log_{23} 1$ (e) $\log_2 \sqrt[7]{32}$

Solution

There are a couple of quick notational issues to deal with first.

$$\ln x = \log_{\mathbf{e}} x \qquad \text{This log is called the natural logarithm}$$
$$\log x = \log_{10} x \qquad \text{This log is called the common logarithm}$$

The **e** in the natural logarithm is the same **e** used in Problem 2 above**.** The common logarithm and the natural logarithm are the logarithms are encountered more often than any other logarithm so the get used to the special notation and special names.

The work required to evaluate the logarithms in this set is the same as in problem in the previous problem.

(a) $\ln \sqrt[3]{\mathbf{e}} = \frac{1}{3}$ because $\mathbf{e}^{\frac{1}{3}} = \sqrt[3]{\mathbf{e}}$

(b) $\log 1000 = 3$ because $10^3 = 1000$

(c) $\log_{16} 16 = 1$ because $16^1 = 16$

(d) $\log_{23} 1 = 0$ because $23^0 = 1$

(e) $\log_2 \sqrt[7]{32} = \frac{5}{7}$ because $\sqrt[7]{32} = 32^{\frac{1}{7}} = \left(2^5\right)^{\frac{1}{7}} = 2^{\frac{5}{7}}$

Logarithm Properties

Complete the following formulas.

1. $\log_b b =$

Solution
$\log_b b = 1$ because $b^1 = b$

2. $\log_b 1 =$

Solution
$\log_b 1 = 0$ because $b^0 = 1$

3. $\log_b b^x =$

Solution
$\log_b b^x = x$

4. $b^{\log_b x} =$

Solution

$b^{\log_b x} = x$

5. $\log_b xy =$

Solution

$\log_b xy = \log_b x + \log_b y$

THERE IS NO SUCH PROPERTY FOR SUMS OR DIFFERENCES!!!!!

$$\log_b(x+y) \neq \log_b x + \log_b y$$
$$\log_b(x-y) \neq \log_b x - \log_b y$$

6. $\log_b\left(\frac{x}{y}\right) =$

Solution

$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

THERE IS NO SUCH PROPERTY FOR SUMS OR DIFFERENCES!!!!!

$$\log_b(x+y) \neq \log_b x + \log_b y$$
$$\log_b(x-y) \neq \log_b x - \log_b y$$

7. $\log_b\left(x^r\right) =$

Solution

$\log_b\left(x^r\right) = r\log_b x$

Note in this case the exponent needs to be on the WHOLE argument of the logarithm. For instance,

$$\log_b(x+y)^2 = 2\log_b(x+y)$$

However,

$$\log_b\left(x^2+y^2\right) \neq 2\log_b(x+y)$$

8. Write down the change of base formula for logarithms.

Solution

Write down the change of base formula for logarithms.

$$\log_b x = \frac{\log_a x}{\log_a b}$$

This is the most general change of base formula and will convert from base b to base a. However, the usual reason for using the change of base formula is so you can compute the value of a logarithm that is in a base that you can't easily compute. Using the change of base formula means that you can write the logarithm in terms of a logarithm that you can compute. The two most common change of base formulas are

$$\log_b x = \frac{\ln x}{\ln b} \qquad \text{and} \qquad \log_b x = \frac{\log x}{\log b}$$

In fact, often you will see one or the other listed as THE change of base formula!

In the problems in the Basic Logarithm Functions section you computed the value of a few logarithms, but you could do these without the change of base formula because all the arguments could be wrote in terms of the base to a power. For instance,

$$\log_7 49 = 2 \qquad \text{because} \qquad 7^2 = 49$$

However, this only works because 49 can be written as a power of 7! We would need the change of base formula to compute $\log_7 50$.

$$\log_7 50 = \frac{\ln 50}{\ln 7} = \frac{3.91202300543}{1.94591014906} = 2.0103821378$$

OR

$$\log_7 50 = \frac{\log 50}{\log 7} = \frac{1.69897000434}{0.845098040014} = 2.0103821378$$

So, it doesn't matter which we use, you will get the same answer regardless.

Note as well that we could use the change of base formula on $\log_7 49$ if we wanted to as well.

$$\log_7 49 = \frac{\ln 49}{\ln 7} = \frac{3.89182029811}{1.94591014906} = 2$$

This is a lot of work however, and is probably not the best way to deal with this.

9. What is the domain of a logarithm?

Solution

The domain $y = \log_b x$ is $x > 0$. In other words you can't plug in zero or a negative number into a logarithm. This makes sense if you remember that $b > 0$ and write the logarithm in exponential form.

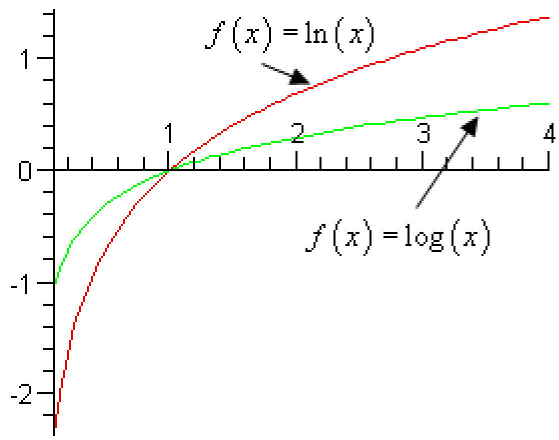
$$y = \log_b x \qquad \Rightarrow \qquad b^y = x$$

Since $b > 0$ there is no way for x to be either zero or negative. Therefore, you can't plug a negative number or zero into a logarithm!

10. Sketch the graph of $f(x) = \ln(x)$ and $g(x) = \log(x)$.

Solution

Not much to this other than to use a calculator to evaluate these at a few points and then make the sketch. Here is the sketch.



From this graph we can see the following behaviors of each graph.

$\ln(x) \to \infty$ as $x \to \infty$ and $\ln(x) \to -\infty$ as $x \to 0$ $(x > 0)$

$\log(x) \to \infty$ as $x \to \infty$ and $\log(x) \to -\infty$ as $x \to 0$ $(x > 0)$

Remember that we require $x > 0$ in each logarithm.

Simplifying Logarithms

Simplify each of the following logarithms.

1. $\ln x^3 y^4 z^5$

Solution

Here simplify means use Property 1 - 7 from the Logarithm Properties section as often as you can. You will be done when you can't use any more of these properties.

Property 5 can be extended to products of more than two functions so,

$$\begin{aligned}\ln x^3 y^4 z^5 &= \ln x^3 + \ln y^4 + \ln z^5 \\ &= 3\ln x + 4\ln y + 5\ln z\end{aligned}$$

2. $\log_3\left(\dfrac{9x^4}{\sqrt{y}}\right)$

Solution
In using property 6 make sure that the logarithm that you subtract is the one that contains the denominator as its argument. Also, note that that I'll be converting the root to exponents in the first step since we'll need that done for a later step.

$$\begin{aligned}\log_3\left(\frac{9x^4}{\sqrt{y}}\right) &= \log_3 9x^4 - \log_3 y^{\frac{1}{2}} \\ &= \log_3 9 + \log_3 x^4 - \log_3 y^{\frac{1}{2}} \\ &= 2 + 4\log_3 x - \frac{1}{2}\log_3 y\end{aligned}$$

Evaluate logs where possible as I did in the first term.

3. $\log\left(\dfrac{x^2+y^2}{(x-y)^3}\right)$

Solution
The point to this problem is mostly the correct use of property 7**.**

$$\begin{aligned}\log\left(\frac{x^2+y^2}{(x-y)^3}\right) &= \log\left(x^2+y^2\right) - \log\left(x-y\right)^3 \\ &= \log\left(x^2+y^2\right) - 3\log\left(x-y\right)\end{aligned}$$

You can use Property 7 on the second term because the WHOLE term was raised to the 3, but in the first logarithm, only the individual terms were squared and not the term as a whole so the 2's must stay where they are!

Solving Exponential Equations

In each of the equations in this section the problem is there is a variable in the exponent. In order to solve these we will need to get the variable out of the exponent. This means using Property 3 and/or 7 from the Logarithm Properties section above. In most cases it will be easier to use Property 3 if possible. So, pick an appropriate logarithm and take the log of both sides, then use Property 3 (or Property 7) where appropriate to simplify. Note that often some simplification will need to be done before taking the logs.

Solve each of the following equations.

1. $2\mathbf{e}^{4x-2} = 9$

Solution

The first thing to note is that Property 3 is $\log_b b^x = x$ and NOT $\log_b\left(2b^x\right) = x$! In other words, we've got to isolate the exponential on one side by itself with a coefficient of 1 (one) before we take logs of both sides.

We'll also need to pick an appropriate log to use. In this case the natural log would be best since the exponential in the problem is $\mathbf{e}^{4x-2}$. So, first isolate the exponential on one side.

$$2\mathbf{e}^{4x-2} = 9$$
$$\mathbf{e}^{4x-2} = \frac{9}{2}$$

Now, take the natural log of both sides and use Property 3 to simplify.

$$\ln\left(\mathbf{e}^{4x-2}\right) = \ln\left(\frac{9}{2}\right)$$
$$4x-2 = \ln\left(\frac{9}{2}\right)$$

Now you all can solve $4x-2=9$ so you can solve the equation above. All you need to remember is that $\ln\left(\frac{9}{2}\right)$ is just a number, just as 9 is a number. So add 2 to both sides, then divide by 4 (or multiply by $\frac{1}{4}$).

$$4x = 2+\ln\left(\frac{9}{2}\right)$$
$$x = \frac{1}{4}\left(2+\ln\left(\frac{9}{2}\right)\right)$$
$$x = 0.8760193492$$

Note that while the natural logarithm was the easiest (since the left side simplified down nicely) we could have used any other log had we wanted to. For instance we could have used the common log as follows. Remember that in this case we won't be able to use Property 3 as this requires both the log and the exponential to have the same base which won't be the case here. Therefore, we'll need to use Property 7 to do the simplification.

$$\log\left(\mathbf{e}^{4x-2}\right) = \log\left(\frac{9}{2}\right)$$

$$\left(4x-2\right)\log\mathbf{e} = \log\left(\frac{9}{2}\right)$$

As you can see the problem here is that we've got a $\log\mathbf{e}$ left over after using Property 7. While this can be dealt with using a calculator, it adds a complexity to the problem that should be avoided if at all possible. Solving gives us

$$4x-2 = \frac{\log\left(\frac{9}{2}\right)}{\log\mathbf{e}}$$

$$4x = 2+\frac{\log\left(\frac{9}{2}\right)}{\log\mathbf{e}}$$

$$x = \frac{1}{4}\left(2+\frac{\log\left(\frac{9}{2}\right)}{\log\mathbf{e}}\right)$$

$$x = 0.8760193492$$

2. $10^{t^2-t} = 100$

Solution

Now, in this case it looks like the best logarithm to use is the common logarithm since left hand side has a base of 10. There's no initial simplification to do, so just take the log of both sides and simplify.

$$\log 10^{t^2-t} = \log 100$$

$$t^2 - t = 2$$

At this point, we've just got a quadratic that can be solved

$$t^2 - t - 2 = 0$$

$$\left(t-2\right)\left(t+1\right) = 0$$

So, it looks like the solutions in this case are $t = 2$ and $t = -1$.

As with the last one you could use a different log here, but it would have made the quadratic significantly messier to solve.

3. $7+15\mathbf{e}^{1-3z} = 10$

Solution

There's a little more initial simplification to do here, but other than that it's similar to the first problem in this section.

$$\begin{aligned} 7+15\mathbf{e}^{1-3z} &= 10 \\ 15\mathbf{e}^{1-3z} &= 3 \\ \mathbf{e}^{1-3z} &= \frac{1}{5} \end{aligned}$$

Now, take the log and solve. Again, we'll use the natural logarithm here.

$$\begin{aligned} \ln\left(\mathbf{e}^{1-3z}\right) &= \ln\left(\frac{1}{5}\right) \\ 1-3z &= \ln\left(\frac{1}{5}\right) \\ -3z &= -1+\ln\left(\frac{1}{5}\right) \\ z &= -\frac{1}{3}\left(-1+\ln\left(\frac{1}{5}\right)\right) \\ z &= 0.8698126372 \end{aligned}$$

4. $x - x\mathbf{e}^{5x+2} = 0$

Solution

This one is a little different from the previous problems in this section since it's got x's both in the exponent and out of the exponent. The first step is to factor an x out of both terms.

DO NOT DIVIDE AN *x* FROM BOTH TERMS!!!!

$$\begin{aligned} x - x\mathbf{e}^{5x+2} &= 0 \\ x\left(1-\mathbf{e}^{5x+2}\right) &= 0 \end{aligned}$$

So, it's now a little easier to deal with. From this we can see that we get one of two possibilities.

$$x = 0 \qquad \text{OR}$$

$$1-\mathbf{e}^{5x+2} = 0$$

The first possibility has nothing more to do, except notice that if we had divided both sides by an x we would have missed this one so be careful. In the second possibility we've got a little more to do. This is an equation similar to the first few that we did in this section.

$$\begin{aligned} \mathbf{e}^{5x+2} &= 1 \\ 5x+2 &= \ln 1 \\ 5x+2 &= 0 \\ x &= -\frac{2}{5} \end{aligned}$$

Don't forget that $\ln 1 = 0$!

So, the two solutions are $x = 0$ and $x = -\frac{2}{5}$.

Solving Logarithm Equations

Solving logarithm equations are similar to exponential equations. First, we isolate the logarithm on one side by itself with a coefficient of one. Then we use Property 4 from the Logarithm Properties section with an appropriate choice of *b*. In choosing the appropriate *b*, we need to remember that the *b* MUST match the base on the logarithm!

Solve each of the following equations.

1. $4\log(1-5x) = 2$

Solution

The first step is to divide by the 4, then we'll convert to an exponential equation.

$$\log(1-5x) = \frac{1}{2}$$
$$10^{\log(1-5x)} = 10^{\frac{1}{2}}$$
$$1-5x = 10^{\frac{1}{2}}$$

Note that since we had a common log in the original equation we were forced to use a base of 10 in the exponential equation. Once we've used Property 4 to simplify the equation we've got an equation that can be solved.

$$1-5x = 10^{\frac{1}{2}}$$
$$-5x = -1+10^{\frac{1}{2}}$$
$$x = -\frac{1}{5}\left(-1+10^{\frac{1}{2}}\right)$$
$$x = -0.4324555320$$

Now, with exponential equations we were done at this point, but we've got a little more work to do in this case. Recall the answer to the domain of a logarithm (the answer to Problem 9 in the Logarithm Properties section). We can't take the logarithm of a negative number or zero.

This does not mean that $x = -0.4324555320$ can't be a solution just because it's negative number! The question we've got to ask is this : does this solution produce a negative number (or zero) when we plug it into the logarithms in the original equation. In other words, is $1-5x$ negative or zero if we plug $x = -0.4324555320$ into it? Clearly, (I hope…) $1-5x$ will be positive when we plug $x = -0.4324555320$ in.

Therefore the solution to this is $x = -0.4324555320$.

Note that it is possible for logarithm equations to have no solutions, so if that should happen don't get to excited!

2. $3+2\ln\left(\frac{x}{7}+3\right) = -4$

Solution
There's a little more simplification work to do initially this time, but it's not too bad.

$$\begin{aligned} 2\ln\left(\frac{x}{7}+3\right) &= -7 \\ \ln\left(\frac{x}{7}+3\right) &= -\frac{7}{2} \\ \mathbf{e}^{\ln\left(\frac{x}{7}+3\right)} &= \mathbf{e}^{-\frac{7}{2}} \\ \frac{x}{7}+3 &= \mathbf{e}^{-\frac{7}{2}} \end{aligned}$$

Now, solve this.

$$\begin{aligned} \frac{x}{7}+3 &= \mathbf{e}^{-\frac{7}{2}} \\ \frac{x}{7} &= -3+\mathbf{e}^{-\frac{7}{2}} \\ x &= 7\left(-3+\mathbf{e}^{-\frac{7}{2}}\right) \\ x &= \text{-}20.78861832 \end{aligned}$$

I'll leave it to you to check that $\frac{x}{7}+3$ will be positive upon plugging $x=-20.78861832$ into it and so we've got the solution to the equation.

3. $2\ln\left(\sqrt{x}\right)-\ln\left(1-x\right)=2$

Solution

This one is a little different from the previous two. There are two logarithms in the problem. All we need to do is use Properties 5 – 7 from the Logarithm Properties section to simplify things into a single logarithm then we can proceed as we did in the previous two problems.

The first step is to get coefficients of one in front of both logs.

$$\begin{aligned} 2\ln\left(\sqrt{x}\right)-\ln\left(1-x\right) &= 2 \\ \ln\left(x\right)-\ln\left(1-x\right) &= 2 \end{aligned}$$

Now, use Property 6 from the Logarithm Properties section to combine into the following log.

$$\ln\left(\frac{x}{1-x}\right)=2$$

Finally, exponentiate both sides and solve.

$$\frac{x}{1-x} = \mathbf{e}^2$$
$$x = \mathbf{e}^2\left(1-x\right)$$
$$x = \mathbf{e}^2 - \mathbf{e}^2 x$$
$$x\left(1+\mathbf{e}^2\right) = \mathbf{e}^2$$
$$x = \frac{\mathbf{e}^2}{1+\mathbf{e}^2}$$
$$x = 0.8807970780$$

Finally, we just need to make sure that the solution, $x = 0.8807970780$, doesn't produce negative numbers in both of the original logarithms. It doesn't, so this is in fact our solution to this problem.

4. $\log x + \log\left(x-3\right) = 1$

Solution

This one is the same as the last one except we'll use Property 5 to do the simplification instead.

$$\log x + \log\left(x-3\right) = 1$$
$$\log\left(x\left(x-3\right)\right) = 1$$
$$10^{\log\left(x^2-3x\right)} = 10^1$$
$$x^2 - 3x = 10$$
$$x^2 - 3x - 10 = 0$$
$$\left(x-5\right)\left(x+2\right) = 0$$

So, potential solutions are $x = 5$ and $x = -2$. Note, however that if we plug $x = -2$ into either of the two original logarithms we would get negative numbers so this can't be a solution. We can however, use $x = 5$.

Therefore, the solution to this equation is $x = 5$.

It is important to check your potential solutions in the **original** equation. If you check them in the second logarithm above (after we've combined the two logs) both solutions will appear to work! This is because in combining the two logarithms we've actually changed the problem. In fact, it is this change that introduces the extra solution that we couldn't use!

So, be careful in finding solutions to equations containing logarithms. Also, do not get locked into the idea that you will get two potential solutions and only one of these will work. It is possible to have problems where both are solutions and where neither are solutions.