Algebra

Exponents

Simplify each of the following as much as possible.

1. $2x^4 y^{-3} x^{-19} + y^{\frac{1}{3}} y^{-\frac{3}{4}}$

2. $x^{\frac{3}{5}} x^2 x^{-\frac{1}{2}}$

3. $\dfrac{x x^{-\frac{1}{3}}}{2x^5}$

4. $\left(\dfrac{2x^{-2} x^{\frac{4}{5}} y^6}{x+y} \right)^{-3}$

5. $\left(\dfrac{x^{\frac{4}{7}} x^{\frac{9}{2}} x^{-\frac{10}{3}} - x^2 x^{-9} x^{\frac{1}{2}}}{x+1} \right)^0$

Absolute Value

1. Evaluate $|5|$ and $|-123|$

2. Eliminate the absolute value bars from $|3-8x|$

3. List as many of the properties of absolute value as you can.

Radicals

Evaluate the following.

1. $\sqrt[3]{125}$

2. $\sqrt[6]{64}$

3. $\sqrt[5]{-243}$

4. $\sqrt[2]{100}$

5. $\sqrt[4]{-16}$

Convert each of the following to exponential form.

6. $\sqrt{7x}$

7. $\sqrt[5]{x^2}$

8. $\sqrt[3]{4x+8}$

Simplify each of the following.

9. $\sqrt[3]{16x^6y^{13}}$ Assume that $x \geq 0$ and $y \geq 0$ for this problem.

10. $\sqrt[4]{16x^8y^{15}}$

Rationalizing

Rationalize each of the following.

1. $\dfrac{3xy}{\sqrt{x}+\sqrt{y}}$

2. $\dfrac{\sqrt{t+2}-2}{t^2-4}$

Functions

1. Given $f(x)=-x^2+6x-11$ and $g(x)=\sqrt{4x-3}$ find each of the following.

   (a) $f(2)$ (b) $g(2)$ (c) $f(-3)$ (d) $g(10)$

   (e) $f(t)$ (f) $f(t-3)$ (g) $f(x-3)$ (h) $f(4x-1)$

2. Given $f(x)=10$ find each of the following.

   (a) $f(7)$ (b) $f(0)$ (c) $f(-14)$

3. Given $f(x)=3x^2-x+10$ and $g(x)=1-20x$ find each of the following.

   (a) $(f-g)(x)$ (b) $\left(\dfrac{f}{g}\right)(x)$ (c) $(fg)(x)$

   (d) $(f\circ g)(5)$ (e) $(f\circ g)(x)$ (f) $(g\circ f)(x)$

Multiplying Polynomials

Multiply each of the following.

1. $(7x-4)(7x+4)$

2. $(2x-5)^2$

3. $2(x+3)^2$

4. $(2x^3-x)\left(\sqrt{x}+\dfrac{2}{x}\right)$

5. $(3x+2)(x^2-9x+12)$

Factoring

Factor each of the following as much as possible.

1. $100x^2 - 81$

2. $100x^2 + 81$

3. $3x^2 + 13x - 10$

4. $25x^2 + 10x + 1$

5. $4x^5 - 8x^4 - 32x^3$

6. $125x^3 - 8$

Simplifying Rational Expressions

Simplify each of the following rational expressions.

1. $\dfrac{2x^2 - 8}{x^2 - 4x + 4}$

2. $\dfrac{x^2 - 5x - 6}{6x - x^2}$

Graphing and Common Graphs

Sketch the graph of each of the following.

1. $y = -\dfrac{2}{5}x + 3$

2. $y = (x+3)^2 - 1$

3. $y = -x^2 + 2x + 3$

4. $f(x) = -x^2 + 2x + 3$

5. $x = y^2 - 6y + 5$

6. $x^2 + \left(y+5\right)^2 = 4$

7. $x^2 + 2x + y^2 - 8y + 8 = 0$

8. $\dfrac{\left(x-2\right)^2}{9} + 4\left(y+2\right)^2 = 1$

9. $\dfrac{\left(x+1\right)^2}{9} - \dfrac{\left(y-2\right)^2}{4} = 1$

10. $y = \sqrt{x}$

11. $y = x^3$

12. $y = |x|$

Solving Equations, Part I

Solve each of the following equations.

1. $x^3 - 3x^2 = x^2 + 21x$

2. $3x^2 - 16x + 1 = 0$

3. $x^2 - 8x + 21 = 0$

Solving Equations, Part II

Solve each of the following equations for *y*.

1. $x = \dfrac{2y-5}{6-7y}$

2. $3x^2\left(3-5y\right) + \sin x = 3xy + 8$

3. $2x^2 + 2y^2 = 5$

Solving Systems of Equations

1. Solve the following system of equations. Interpret the solution.

$$\begin{aligned} 2x - y - 2z &= -3 \\ x + 3y + z &= -1 \\ 5x - 4y + 3z &= 10 \end{aligned}$$

2. Determine where the following two curves intersect.

$$\begin{aligned} &x^2 + y^2 = 13 \\ &y = x^2 - 1 \end{aligned}$$

3. Graph the following two curves and determine where they intersect.

$$\begin{aligned} &x = y^2 - 4y - 8 \\ &x = 5y + 28 \end{aligned}$$

Solving Inequalities

Solve each of the following inequalities.

1. $x^2 - 10 > 3x$

2. $x^4 + 4x^3 - 12x^2 \le 0$

3. $3x^2 - 2x - 11 > 0$

4. $\dfrac{x-3}{x+2} \ge 0$

5. $\dfrac{x^2 - 3x - 10}{x-1} < 0$

6. $\dfrac{2x}{x+1} \ge 3$

Absolute Value Equations and Inequalities

Solve each of the following.

1. $\left|3x + 8\right| = 2$

2. $|2x-4|=10$

3. $|x+1|=-15$

4. $|7x-10|\le 4$

5. $|1-2x|<7$

6. $|x-9|\le -1$

7. $|4x+5|>3$

8. $|4-11x|\ge 9$

9. $|10x+1|>-4$

Trigonometry

Trig Function Evaluation

One of the problems with most trig classes is that they tend to concentrate on right triangle trig and do everything in terms of degrees. Then you get to a calculus course where almost everything is done in radians and the unit circle is a very useful tool.

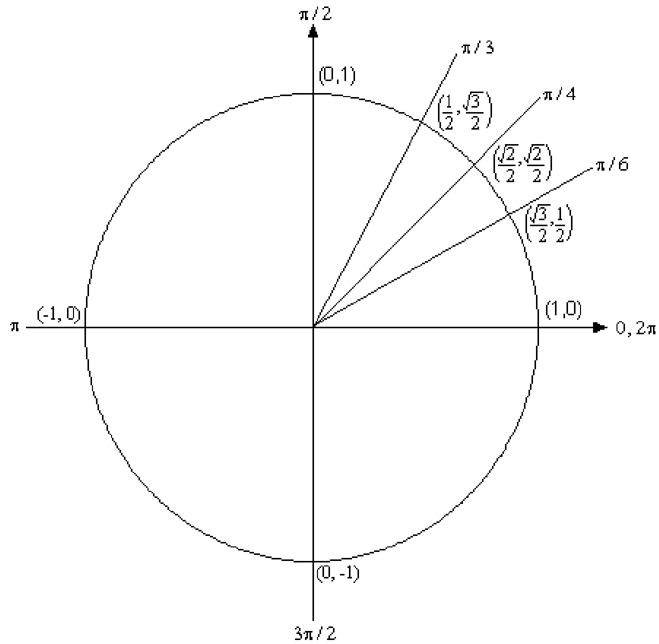
So first off let's look at the following table to relate degrees and radians.

Degree	0	30	45	60	90	180	270	360
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Know this table! There are, of course, many other angles in radians that we'll see during this class, but most will relate back to these few angles. So, if you can deal with these angles you will be able to deal with most of the others.

Be forewarned, everything in most calculus classes will be done in radians!

Now, let's look at the unit circle. Below is the unit circle with just the first quadrant filled in. The way the unit circle works is to draw a line from the center of the circle outwards corresponding to a given angle. Then look at the coordinates of the point where the line and the circle intersect. The first coordinate is the cosine of that angle and the second coordinate is the sine of that angle. There are a couple of *basic* angles that are commonly used. These are $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π and are shown below along with the coordinates of the intersections. So, from the unit circle below we can see that $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.



Remember how the signs of angles work. If you rotate in a counter clockwise direction the angle is positive and if you rotate in a clockwise direction the angle is negative.

Recall as well that one complete revolution is 2π, so the positive x-axis can correspond to either an angle of 0 or 2π (or 4π, or 6π, or -2π, or -4π, *etc.* depending on the direction of rotation). Likewise, the angle $\frac{\pi}{6}$ (to pick an angle completely at random) can also be any of the following angles:

$$\frac{\pi}{6} + 2\pi = \frac{13\pi}{6} \quad \text{(start at } \frac{\pi}{6} \text{ then rotate once around counter clockwise)}$$

$$\frac{\pi}{6} + 4\pi = \frac{25\pi}{6} \quad \text{(start at } \frac{\pi}{6} \text{ then rotate around twice counter clockwise)}$$

$$\frac{\pi}{6} - 2\pi = -\frac{11\pi}{6} \quad \text{(start at } \frac{\pi}{6} \text{ then rotate once around clockwise)}$$

$$\frac{\pi}{6} - 4\pi = -\frac{23\pi}{6} \text{ (start at } \frac{\pi}{6} \text{ then rotate around twice clockwise)}$$

etc.

In fact $\frac{\pi}{6}$ can be any of the following angles $\frac{\pi}{6} + 2\pi\, n$, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$ In this case n is the number of complete revolutions you make around the unit circle starting at $\frac{\pi}{6}$. Positive values of n correspond to counter clockwise rotations and negative values of n correspond to clockwise rotations.

So, why did I only put in the first quadrant? The answer is simple. If you know the first quadrant then you can get all the other quadrants from the first. You'll see this in the following examples.

Find the exact value of each of the following. In other words, don't use a calculator.

10. $\sin\left(\frac{2\pi}{3}\right)$ and $\sin\left(-\frac{2\pi}{3}\right)$

11. $\cos\left(\frac{7\pi}{6}\right)$ and $\cos\left(-\frac{7\pi}{6}\right)$

12. $\tan\left(-\frac{\pi}{4}\right)$ and $\tan\left(\frac{7\pi}{4}\right)$

13. $\sin\left(\frac{9\pi}{4}\right)$

14. $\sec\left(\frac{25\pi}{6}\right)$

15. $\tan\left(\frac{4\pi}{3}\right)$

Trig Evaluation Final Thoughts

As we saw in the previous examples if you know the first quadrant of the unit circle you can find the value of ANY trig function (not just sine and cosine) for ANY angle that can be related back to one of those shown in the first quadrant. This is a nice idea to remember as it means that you only need to memorize the first quadrant and how to get the angles in the remaining three quadrants!

In these problems I used only "basic" angles, but many of the ideas here can also be applied to angles other than these "basic" angles as we'll see in Solving Trig Equations.

Graphs of Trig Functions

There is not a whole lot to this section. It is here just to remind you of the graphs of the six trig functions as well as a couple of nice properties about trig functions.

Before jumping into the problems remember we saw in the Trig Function Evaluation section that trig functions are examples of *periodic* functions. This means that all we really need to do is graph the function for one periods length of values then repeat the graph.

Graph the following function.

1. $y = \cos(x)$

2. $y = \cos(2x)$

3. $y = 5\cos(2x)$

4. $y = \sin(x)$

5. $y = \sin\left(\frac{x}{3}\right)$

6. $y = \tan(x)$

7. $y = \sec(x)$

8. $y = \csc(x)$

9. $y = \cot(x)$

Trig Formulas

This is not a complete list of trig formulas. This is just a list of formulas that I've found to be the most useful in a Calculus class. For a complete listing of trig formulas you can download my Trig Cheat Sheet.

Complete the following formulas.

1. $\sin^2(\theta)+\cos^2(\theta)=$

2. $\tan^2(\theta)+1=$

3. $\sin(2t)=$

4. $\cos(2x)=$ (Three possible formulas)

5. $\cos^2(x)=$ (In terms of cosine to the first power)

6. $\sin^2(x)=$ (In terms of cosine to the first power)

Solving Trig Equations

Solve the following trig equations. For those without intervals listed find ALL possible solutions. For those with intervals listed find only the solutions that fall in those intervals.

1. $2\cos(t)=\sqrt{3}$

2. $2\cos(t)=\sqrt{3}$ on $[-2\pi,2\pi]$

3. $2\sin(5x)=-\sqrt{3}$

4. $2\sin(5x+4)=-\sqrt{3}$

5. $2\sin(3x)=1$ on $[-\pi,\pi]$

6. $\sin(4t)=1$ on $[0,2\pi]$

7. $\cos(3x) = 2$

8. $\sin(2x) = -\cos(2x)$

9. $2\sin(\theta)\cos(\theta) = 1$

10. $\sin(w)\cos(w) + \cos(w) = 0$

11. $2\cos^2(3x) + 5\cos(3x) - 3 = 0$

12. $5\sin(2x) = 1$

13. $4\cos\left(\frac{x}{5}\right) = -3$

14. $10\sin(x-2) = -7$

Inverse Trig Functions

One of the more common notations for inverse trig functions can be very confusing. First, regardless of how you are used to dealing with exponentiation we tend to denote an inverse trig function with an "exponent" of "-1". In other words, the inverse cosine is denoted as $\cos^{-1}(x)$. It is important here to note that in this case the "-1" is NOT an exponent and so,

$$\cos^{-1}(x) \neq \frac{1}{\cos(x)}.$$

In inverse trig functions the "-1" looks like an exponent but it isn't, it is simply a notation that we use to denote the fact that we're dealing with an inverse trig function. It is a notation that we use in this case to denote inverse trig functions. If I had really wanted exponentiation to denote 1 over cosine I would use the following.

$$\left(\cos(x)\right)^{-1} = \frac{1}{\cos(x)}$$

There's another notation for inverse trig functions that avoids this ambiguity. It is the following.

$$\begin{aligned} \cos^{-1}(x) &= \arccos(x) \\ \sin^{-1}(x) &= \arcsin(x) \\ \tan^{-1}(x) &= \arctan(x) \end{aligned}$$

So, be careful with the notation for inverse trig functions!

There are, of course, similar inverse functions for the remaining three trig functions, but these are the main three that you'll see in a calculus class so I'm going to concentrate on them.

To evaluate inverse trig functions remember that the following statements are equivalent.

$$\theta = \cos^{-1}(x) \qquad \Leftrightarrow \qquad x = \cos(\theta)$$
$$\theta = \sin^{-1}(x) \qquad \Leftrightarrow \qquad x = \sin(\theta)$$
$$\theta = \tan^{-1}(x) \qquad \Leftrightarrow \qquad x = \tan(\theta)$$

In other words, when we evaluate an inverse trig function we are asking what angle, θ, did we plug into the trig function (regular, not inverse!) to get x.

So, let's do some problems to see how these work. Evaluate each of the following.

1. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

2. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

3. $\sin^{-1}\left(-\frac{1}{2}\right)$

4. $\tan^{-1}(1)$

5. $\cos\left(\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$

6. $\sin^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right)$

7. $\tan\left(\tan^{-1}(-4)\right)$.

Exponentials / Logarithms

Basic Exponential Functions

First, let's recall that for $b > 0$ and $b \neq 1$ an exponential function is any function that is in the form

$$f(x) = b^x$$

We require $b \neq 1$ to avoid the following situation,

$$f(x) = 1^x = 1$$

So, if we allowed $b \neq 1$ we would just get the constant function, 1.

We require $b > 0$ to avoid the following situation,

$$f(x) = (-4)^x \qquad \Rightarrow \qquad f\left(\frac{1}{2}\right) = (-4)^{\frac{1}{2}} = \sqrt{-4}$$

By requiring $b > 0$ we don't have to worry about the possibility of square roots of negative numbers.

1. Evaluate $f(x) = 4^x$, $g(x) = \left(\frac{1}{4}\right)^x$ and $h(x) = 4^{-x}$ at $x = -2, -1, 0, 1, 2$.

2. Sketch the graph of $f(x) = 4^x$, $g(x) = \left(\frac{1}{4}\right)^x$ and $h(x) = 4^{-x}$ on the same axis system.

3. List as some basic properties for $f(x) = b^x$.

4. Evaluate $f(x) = \mathbf{e}^x$, $g(x) = \mathbf{e}^{-x}$ and $h(x) = 5\mathbf{e}^{1-3x}$ at $x = -2, -1, 0, 1, 2$.

5. Sketch the graph of $f(x) = \mathbf{e}^x$ and $g(x) = \mathbf{e}^{-x}$.

Basic Logarithmic Functions

1. Without a calculator give the exact value of each of the following logarithms.

   (a) $\log_2 16$ (b) $\log_4 16$ (c) $\log_5 625$

   (d) $\log_9 \frac{1}{531441}$ (e) $\log_{\frac{1}{6}} 36$ (f) $\log_{\frac{3}{2}} \frac{27}{8}$

2. Without a calculator give the exact value of each of the following logarithms.

   (a) $\ln \sqrt[3]{\mathbf{e}}$ (b) $\log 1000$ (c) $\log_{16} 16$

(d) $\log_{23} 1$ (e) $\log_2 \sqrt[7]{32}$

Logarithm Properties

Complete the following formulas.

1. $\log_b b =$

2. $\log_b 1 =$

3. $\log_b b^x =$

4. $b^{\log_b x} =$

5. $\log_b xy =$

6. $\log_b \left(\frac{x}{y}\right) =$

7. $\log_b \left(x^r\right) =$

8. Write down the change of base formula for logarithms.

9. What is the domain of a logarithm?

10. Sketch the graph of $f(x) = \ln(x)$ and $g(x) = \log(x)$.

Simplifying Logarithms

Simplify each of the following logarithms.

1. $\ln x^3 y^4 z^5$

2. $\log_3 \left(\frac{9x^4}{\sqrt{y}}\right)$

3. $\log\left(\dfrac{x^2+y^2}{(x-y)^3}\right)$

Solving Exponential Equations

In each of the equations in this section the problem is there is a variable in the exponent. In order to solve these we will need to get the variable out of the exponent. This means using Property 3 and/or 7 from the Logarithm Properties section above. In most cases it will be easier to use Property 3 if possible. So, pick an appropriate logarithm and take the log of both sides, then use Property 3 (or Property 7) where appropriate to simplify. Note that often some simplification will need to be done before taking the logs.

Solve each of the following equations.

1. $2\mathbf{e}^{4x-2}=9$

2. $10^{t^2-t}=100$

3. $7+15\mathbf{e}^{1-3z}=10$

4. $x-x\mathbf{e}^{5x+2}=0$

Solving Logarithm Equations

Solving logarithm equations are similar to exponential equations. First, we isolate the logarithm on one side by itself with a coefficient of one. Then we use Property 4 from the Logarithm Properties section with an appropriate choice of b. In choosing the appropriate b, we need to remember that the b MUST match the base on the logarithm!

Solve each of the following equations.

1. $4\log(1-5x)=2$

2. $3+2\ln\left(\dfrac{x}{7}+3\right)=-4$

3. $2\ln\left(\sqrt{x}\right) - \ln\left(1-x\right) = 2$

4. $\log x + \log\left(x-3\right) = 1$