Limits
Definitions

**Precise Definition:** We say \( \lim_{x \to a} f(x) = L \) if for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that whenever \( 0 < |x - a| < \delta \) then \( |f(x) - L| < \varepsilon \).

**“Working” Definition:** We say \( \lim_{x \to a} f(x) = L \) if we can make \( f(x) \) as close to \( L \) as we want by taking \( x \) sufficiently close to \( a \) (on either side of \( a \)) without letting \( x = a \).

**Right hand limit:** \( \lim_{x \to a^+} f(x) = L \). This has the same definition as the limit except it requires \( x > a \).

**Left hand limit:** \( \lim_{x \to a^-} f(x) = L \). This has the same definition as the limit except it requires \( x < a \).

**Limit at Infinity:** We say \( \lim_{x \to \infty} f(x) = L \) if we can make \( f(x) \) as close to \( L \) as we want by taking \( x \) large enough and positive.

There is a similar definition for \( \lim_{x \to -\infty} f(x) = L \) except we require \( x \) large and negative.

**Infinite Limit:** We say \( \lim_{x \to a} f(x) = \infty \) if we can make \( f(x) \) arbitrarily large (and positive) by taking \( x \) sufficiently close to \( a \) (on either side of \( a \)) without letting \( x = a \).

There is a similar definition for \( \lim_{x \to a} f(x) = -\infty \) except we make \( f(x) \) arbitrarily large and negative.

**Relationship between the limit and one-sided limits**

\[
\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x) \Rightarrow \lim_{x \to a} f(x) \text{ Does Not Exist}
\]

**Properties**
Assume \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) both exist and \( c \) is any number then,

1. \( \lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x) \)
2. \( \lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) \)
3. \( \lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x) \)
4. \( \lim_{x \to a} \left[ \frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \) provided \( \lim_{x \to a} g(x) \neq 0 \)
5. \( \lim_{x \to a} \left[ f(x) \right]^n = \left[ \lim_{x \to a} f(x) \right]^n \)
6. \( \lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} \)

**Basic Limit Evaluations at \( \pm \infty \)**

Note: \( \text{sgn}(a) = 1 \) if \( a > 0 \) and \( \text{sgn}(a) = -1 \) if \( a < 0 \).

1. \( \lim_{x \to \infty} e^x = \infty \) & \( \lim_{x \to -\infty} e^x = 0 \)
2. \( \lim_{x \to \infty} \ln(x) = \infty \) & \( \lim_{x \to 0^+} \ln(x) = -\infty \)
3. If \( r > 0 \) then \( \lim_{x \to 0} \frac{b}{x^r} = 0 \)
4. If \( r > 0 \) and \( x^r \) is real for negative \( x \) then \( \lim_{x \to -\infty} \frac{b}{x^r} = 0 \)
5. \( n \) even: \( \lim_{x \to \pm \infty} x^n = \infty \)
6. \( n \) odd: \( \lim_{x \to \pm \infty} x^n = -\infty \)
7. \( n \) even: \( \lim_{x \to \pm \infty} ax^n + \cdots + bx + c = \text{sgn}(a)\infty \)
8. \( n \) odd: \( \lim_{x \to \pm \infty} ax^n + \cdots + bx + c = \text{sgn}(a)\infty \)
9. \( n \) odd: \( \lim_{x \to -\infty} ax^n + \cdots + cx + d = -\text{sgn}(a)\infty \)
**Evaluation Techniques**

**Continuous Functions**

If \( f(x) \) is continuous at \( a \) then \( \lim_{x \to a} f(x) = f(a) \)

**Continuous Functions and Composition**

\( f(x) \) is continuous at \( b \) and \( \lim_{x \to a} g(x) = b \) then

\[ \lim_{x \to a} f(g(x)) = f \left( \lim_{x \to a} g(x) \right) = f(b) \]

**Factor and Cancel**

\[
\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x} = \lim_{x \to 2} \frac{(x-2)(x+6)}{x(x-2)}
\]

\[ = \lim_{x \to 2} \frac{x+6}{x} = \frac{8}{2} = 4 \]

**Rationalize Numerator/Denominator**

\[
\lim_{x \to 9} \frac{3 - \sqrt{x}}{x^2 - 81} = \lim_{x \to 9} \frac{3 - \sqrt{x}}{3 + \sqrt{x}} \cdot \frac{3 + \sqrt{x}}{3 + \sqrt{x}}
\]

\[ = \lim_{x \to 9} \frac{9 - x}{(x^2 - 81)(3 + \sqrt{x})} = \lim_{x \to 9} \frac{9 - x}{(x+9)(3 + \sqrt{x})}
\]

\[ = \frac{-1}{18} = -\frac{1}{108} \]

**Combine Rational Expressions**

\[
\lim_{h \to 0} \frac{1}{h} \left( \frac{-x}{h} - \frac{1}{x} \right) = \lim_{h \to 0} \frac{1}{h} \left( \frac{x - (x+h)}{x(x+h)} \right)
\]

\[ = \lim_{h \to 0} \frac{1}{h} \left( \frac{-h}{x(x+h)} \right) = \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2} \]

**L’Hospital’s Rule**

If \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \) or \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\pm \infty}{\pm \infty} \) then,

\[ \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \]

\( a \) is a number, \( \infty \) or \( -\infty \)

**Polynomials at Infinity**

\( p(x) \) and \( q(x) \) are polynomials. To compute

\[ \lim_{x \to \pm\infty} \frac{p(x)}{q(x)} \]

factor largest power of \( x \) in \( q(x) \) out of both \( p(x) \) and \( q(x) \) then compute limit.

\[
\lim_{x \to \pm\infty} \frac{3x^2 - 4}{5x - 2x^2} = \lim_{x \to \pm\infty} \frac{x^2 \left( \frac{3 - 4}{x^2} \right)}{x^2 \left( \frac{5}{x^2} - 2 \right)} = \lim_{x \to \pm\infty} \frac{3 - 4}{x^2} = -\frac{3}{2}
\]

**Piecewise Function**

\[ \lim_{x \to 2^-} \log(x) \text{ where } g(x) = \begin{cases} x^2 + 5 & \text{if } x < -2 \\ 1 - 3x & \text{if } x \geq -2 \end{cases} \]

Compute two one sided limits,

\[ \lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} x^2 + 5 = 9 \]

\[ \lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} 1 - 3x = 7 \]

One sided limits are different so \( \lim_{x \to 2^-} g(x) \) doesn’t exist. If the two one sided limits had been equal then \( \lim_{x \to 2^-} g(x) \) would have existed and had the same value.

**Some Continuous Functions**

Partial list of continuous functions and the values of \( x \) for which they are continuous.

1. Polynomials for all \( x \).
2. Rational function, except for \( x \)'s that give division by zero.
3. \( \sqrt[3]{x} \) (\( n \) odd) for all \( x \).
4. \( \sqrt[2]{x} \) (\( n \) even) for all \( x \geq 0 \).
5. \( e^x \) for all \( x \).
6. \( \ln x \) for \( x > 0 \).
7. \( \cos(x) \) and \( \sin(x) \) for all \( x \).
8. \( \tan(x) \) and \( \sec(x) \) provided \( x \neq \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \)
9. \( \cot(x) \) and \( \csc(x) \) provided \( x \neq \ldots, -2\pi, -\pi, \pi, 2\pi, \ldots \)

**Intermediate Value Theorem**

Suppose that \( f(x) \) is continuous on \([a, b]\) and let \( M \) be any number between \( f(a) \) and \( f(b) \).

Then there exists a number \( c \) such that \( a < c < b \) and \( f(c) = M \).
**Derivatives**

**Definition and Notation**

If \( y = f(x) \) then the derivative is defined to be
\[
Df(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.
\]

If \( y = f(x) \) then all of the following are equivalent notations for the derivative.

- \( y' = \frac{dy}{dx} = \frac{df}{dx} \)
- \( f'(x) = \frac{dy}{dx} \bigg|_{x=a} = \frac{df}{dx} \bigg|_{x=a} = \frac{dy}{dx} \)

**Interpretation of the Derivative**

If \( y = f(x) \) then,

1. \( m = f'(a) \) is the slope of the tangent line to \( y = f(x) \) at \( x = a \) and the equation of the tangent line at \( x = a \) is given by 
\[
y = f(a) + f'(a)(x - a).
\]
2. \( f'(a) \) is the instantaneous rate of change of \( f(x) \) at \( x = a \).
3. If \( f(x) \) is the position of an object at time \( x \) then \( f'(a) \) is the velocity of the object at \( x = a \).

**Basic Properties and Formulas**

If \( f(x) \) and \( g(x) \) are differentiable functions (the derivative exists), \( c \) and \( n \) are any real numbers,

1. \( (c f)' = c f'(x) \) \hspace{1cm} 5. \( \frac{d}{dx}(c) = 0 \)
2. \( (f \pm g)' = f'(x) \pm g'(x) \) \hspace{1cm} 6. \( \frac{d}{dx}(x^n) = nx^{n-1} \) \text{ – Power Rule}
3. \( (f g)' = f'g + fg' \) \text{ – Product Rule} \hspace{1cm} 7. \( \frac{d}{dx}(f(g(x))) = f'(g(x))g'(x) \)
4. \( \left( \frac{f}{g} \right)' = \frac{f'g - fg'}{g^2} \) \text{ – Quotient Rule} \hspace{1cm} \text{This is the Chain Rule}

**Common Derivatives**

- \( \frac{d}{dx}(x) = 1 \)
- \( \frac{d}{dx} \sin x = \cos x \)
- \( \frac{d}{dx} \cos x = -\sin x \)
- \( \frac{d}{dx} \tan x = \sec^2 x \)
- \( \frac{d}{dx} \sec x = \sec x \tan x \)
- \( \frac{d}{dx} \csc x = -\csc x \cot x \)
- \( \frac{d}{dx} \cot x = -\csc^2 x \)
- \( \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \)
- \( \frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}} \)
- \( \frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2} \)
- \( \frac{d}{dx} \ln(x) = \frac{1}{x}, \quad x > 0 \)
- \( \frac{d}{dx} \ln|x| = \frac{1}{x}, \quad x \neq 0 \)
- \( \frac{d}{dx} \log_a(x) = \frac{1}{x \ln a}, \quad x > 0 \)

Visit [http://tutorial.math.lamar.edu](http://tutorial.math.lamar.edu) for a complete set of Calculus notes. © 2005 Paul Dawkins
Chain Rule Variants

The chain rule applied to some specific functions.

1. \[
\frac{d}{dx} \left( f(x)^n \right) = n \left( f(x) \right)^{n-1} f'(x)
\]
2. \[
\frac{d}{dx} (e^{f(x)}) = f'(x) e^{f(x)}
\]
3. \[
\frac{d}{dx} \left( \ln[f(x)] \right) = \frac{f'(x)}{f(x)}
\]
4. \[
\frac{d}{dx} \left( \sin f(x) \right) = f'(x) \cos f(x)
\]
5. \[
\frac{d}{dx} \left( \cos f(x) \right) = -f'(x) \sin f(x)
\]
6. \[
\frac{d}{dx} \left( \tan f(x) \right) = f'(x) \sec^2 f(x)
\]
7. \[
\frac{d}{dx} \left( \sec f(x) \right) = f'(x) \sec f(x) \tan f(x)
\]
8. \[
\frac{d}{dx} \left( \tan^{-1} f(x) \right) = \frac{f'(x)}{1 + [f(x)]^2}
\]

Higher Order Derivatives

The Second Derivative is denoted as \( f''(x) = \frac{d^2 f}{dx^2} \) and is defined as \( f''(x) = \left( f'(x) \right)' \), i.e. the derivative of the first derivative, \( f'(x) \).

The \( n \)th Derivative is denoted as \( f^{(n)}(x) = \frac{d^n f}{dx^n} \) and is defined as \( f^{(n)}(x) = \left( f^{(n-1)}(x) \right)' \), i.e. the derivative of the \((n-1)\)st derivative, \( f^{(n-1)}(x) \).

Implicit Differentiation

Find \( y' \) if \( e^{2x-9y} + x^3 y^2 = \sin(y) + 11x \). Remember \( y = y(x) \) here, so products/quotients of \( x \) and \( y \) will use the product/quotient rule and derivatives of \( y \) will use the chain rule. The “trick” is to differentiate as normal and every time you differentiate a \( y \) you tack on a \( y' \) (from the chain rule). After differentiating solve for \( y' \).

\[
\begin{align*}
e^{2x-9y}(2-9y') + 3x^2 y^2 + 2x^3 y y' &= \cos(y) y' + 11 \\
2e^{2x-9y} - 9y e^{2x-9y} + 3x^2 y^2 + 2x^3 y y' &= \cos(y) y' + 11 \\
(2x^3 y - 9e^{2x-9y} - \cos(y)) y' &= 11 - 2e^{2x-9y} - 3x^2 y^2
\end{align*}
\]

\[
\Rightarrow \quad y' = \frac{11 - 2e^{2x-9y} - 3x^2 y^2}{2x^3 y - 9e^{2x-9y} - \cos(y)}
\]

Increasing/Decreasing – Concave Up/Concave Down

Critical Points
\( x = c \) is a critical point of \( f(x) \) provided either
1. \( f'(c) = 0 \) or 2. \( f'(c) \) doesn’t exist.

Increasing/Decreasing
1. If \( f'(x) > 0 \) for all \( x \) in an interval \( I \) then \( f(x) \) is increasing on the interval \( I \).
2. If \( f'(x) < 0 \) for all \( x \) in an interval \( I \) then \( f(x) \) is decreasing on the interval \( I \).
3. If \( f'(x) = 0 \) for all \( x \) in an interval \( I \) then \( f(x) \) is constant on the interval \( I \).

Concave Up/Concave Down
1. If \( f''(x) > 0 \) for all \( x \) in an interval \( I \) then \( f(x) \) is concave up on the interval \( I \).
2. If \( f''(x) < 0 \) for all \( x \) in an interval \( I \) then \( f(x) \) is concave down on the interval \( I \).

Inflection Points
\( x = c \) is a inflection point of \( f(x) \) if the concavity changes at \( x = c \).
**Extrema**

1. **Absolute Extrema**
   - A point on the graph of a function where the function attains its maximum or minimum value within a given interval.
   - An absolute maximum occurs at a point where the function value is greater than or equal to all other function values in the domain.
   - An absolute minimum occurs at a point where the function value is less than or equal to all other function values in the domain.

2. **Relative (local) Extrema**
   - A point on the graph of a function where the function attains its maximum or minimum value within a given interval.
   - A relative maximum occurs at a point where the function value is greater than or equal to all other function values in the neighborhood of the point.
   - A relative minimum occurs at a point where the function value is less than or equal to all other function values in the neighborhood of the point.

3. **Mean Value Theorem**
   - If a function is continuous on a closed interval and differentiable on the open interval, then there exists at least one point in the interval where the derivative equals the average rate of change of the function over the interval.

4. **Newton’s Method**
   - An iterative method for finding successively better approximations to the roots (or zeroes) of a real-valued function.

---

**Fermat’s Theorem**

- If \( f(x) \) is a function with a relative (or local) extrema at \( x = c \), then \( x = c \) is a critical point of \( f(x) \). A critical point is a point where the derivative is 0 or does not exist.

---

**Extreme Value Theorem**

- If \( f(x) \) is continuous on the closed interval \([a, b]\) and \( f(x) \) is differentiable on the open interval \((a, b)\), then there exists a number \( c \) in \((a, b)\) such that \( f(c) \) is the absolute maximum of \( f(x) \) on \([a, b]\).

---

**Finding Absolute Extrema**

1. Find all critical points of \( f(x) \) in \([a, b]\).
2. Evaluate \( f(x) \) at all points found in Step 1.
3. Evaluate \( f(a) \) and \( f(b) \).
4. Identify the abs. max. (largest function value) and the abs. min. (smallest function value) from the evaluations in Steps 2 & 3.

---

**1st Derivative Test**

- If \( x = c \) is a critical point of \( f(x) \) and \( f(x) \) changes sign from positive to negative as \( x \) increases through \( c \), then \( x = c \) is a relative maximum of \( f(x) \).
- If \( x = c \) is a critical point of \( f(x) \) and \( f(x) \) changes sign from negative to positive as \( x \) increases through \( c \), then \( x = c \) is a relative minimum of \( f(x) \).

---

**2nd Derivative Test**

- If \( x = c \) is a critical point of \( f(x) \) and \( f''(c) > 0 \), then \( x = c \) is a relative minimum of \( f(x) \).
- If \( x = c \) is a critical point of \( f(x) \) and \( f''(c) < 0 \), then \( x = c \) is a relative maximum of \( f(x) \).
- If \( x = c \) is a critical point of \( f(x) \) and \( f''(c) = 0 \), then the test is inconclusive.

---

**Finding Relative Extrema and/or Classify Critical Points**

1. Find all critical points of \( f(x) \).
2. Use the 1st derivative test or the 2nd derivative test on each critical point.

---

Visit [http://tutorial.math.lamar.edu](http://tutorial.math.lamar.edu) for a complete set of Calculus notes. © 2005 Paul Dawkins
Related Rates
Sketch picture and identify known/unknown quantities. Write down equation relating quantities and differentiate with respect to \( t \) using implicit differentiation (i.e. add on a derivative every time you differentiate a function of \( t \)). Plug in known quantities and solve for the unknown quantity.

**Ex.** A 15 foot ladder is resting against a wall. The bottom is initially 10 ft away and is being pushed towards the wall at \( \frac{1}{4} \) ft/sec. How fast is the top moving after 12 sec?

\[
\begin{align*}
x' & = -\frac{1}{4} \\
x & = 10 - 12\left(\frac{1}{4}\right) = 7 \\
y & = \sqrt{15^2 - 7^2} = \sqrt{176}
\end{align*}
\]

\( y' \) is negative because \( x \) is decreasing. Using Pythagorean Theorem and differentiating,
\[
x^2 + y^2 = 15^2 \Rightarrow 2x \cdot x' + 2y \cdot y' = 0
\]
After 12 sec we have \( x = 10 - 12\left(\frac{1}{4}\right) = 7 \) and so \( y = \sqrt{15^2 - 7^2} = \sqrt{176} \). Plug in and solve for \( y' \).

\[
7\left(-\frac{1}{4}\right) + \sqrt{176} \cdot y' = 0 \Rightarrow y' = \frac{7}{4\sqrt{176}} \text{ ft/sec}
\]

**Optimization**
Sketch picture if needed, write down equation to be optimized and constraint. Solve constraint for one of the two variables and plug into first equation. Find critical points of equation in range of variables and verify that they are min/max as needed.

**Ex.** We’re enclosing a rectangular field with 500 ft of fence material and one side of the field is a building. Determine dimensions that will maximize the enclosed area.

Maximize \( A = xy \) subject to constraint of \( x + 2y = 500 \). Solve constraint for \( x \) and plug into area.

\[
x = 500 - 2y \Rightarrow A = y(500 - 2y)
\]

\[
= 500y - 2y^2
\]

Differentiate and find critical point(s).
\[
A' = 500 - 4y \Rightarrow y = 125
\]
By 2\textsuperscript{nd} deriv. test this is a rel. max. and so is the answer we’re after. Finally, find \( x \).

\[
x = 500 - 2(125) = 250
\]

The dimensions are then 250 x 125.
Integrals

Definitions

**Definite Integral:** Suppose \( f(x) \) is continuous on \([a,b]\). Divide \([a,b]\) into \(n\) subintervals of width \( \Delta x \) and choose \( x_i^* \) from each interval.

Then \( \int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x \).

**Anti-Derivative:** An anti-derivative of \( f(x) \) is a function, \( F(x) \), such that \( F'(x) = f(x) \).

**Indefinite Integral:** \( \int f(x) \, dx = F(x) + c \)

where \( F(x) \) is an anti-derivative of \( f(x) \).

Fundamental Theorem of Calculus

**Part I:** If \( f(x) \) is continuous on \([a,b]\) then

\[
\int_a^b f(x) \, dx = F(b) - F(a).
\]

**Variants of Part I:**

\[
\int_a^b f(x) \, dx = \frac{d}{dx} \int_a^b u(x) \, f(t) \, dt = u'(x) f[u(x)] - v'(x) f[v(x)],
\]

\[
\int_a^b f(x) \, dx = \int_a^b u'(x) f[u(x)] - v'(x) f[v(x)].
\]

**Properties**

\[
\begin{align*}
\int_a^b f(x) \pm g(x) \, dx &= \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx \\
\int_a^b f(x) \, dx &= \int_a^b g(x) \, dx \quad \text{if} \quad f(x) = g(x) \quad \text{on} \quad [a,b] \\
\int_a^b f(x) \, dx &= \left[ F(x) \right]_a^b = F(b) - F(a) \\
\int_a^b c \, dx &= c \int_a^b f(x) \, dx \quad \text{if} \quad c \text{ is a constant}
\end{align*}
\]

Common Integrals

\[
\begin{align*}
\int k \, dx &= k \, x + c \\
\int x^n \, dx &= \frac{1}{n+1} x^{n+1} + c, \quad n \neq -1 \\
\int x^{-1} \, dx &= \ln |x| + c \\
\int \frac{1}{ax+b} \, dx &= \frac{1}{a} \ln |ax+b| + c \\
\int \ln u \, du &= u \ln (u) - u + c \\
\int e^{u} \, du &= e^{u} + c \\
\int \cos u \, du &= \sin u + c \\
\int \sec^2 u \, du &= \tan u + c \\
\int \sec u \tan u \, du &= \sec u + c \\
\int \csc u \cot u \, du &= -\csc u + c \\
\int \tan u \, du &= \ln |\sec u| + c \\
\int \sec u \, du &= \ln |\sec u + \tan u| + c \\
\int \frac{1}{a^2 + u^2} \, du &= \frac{1}{a} \tan^{-1} \left( \frac{u}{a} \right) + c \\
\int \frac{1}{\sqrt{a^2 - u^2}} \, du &= \sin^{-1} \left( \frac{u}{a} \right) + c
\end{align*}
\]


**Calculus Cheat Sheet**

**Standard Integration Techniques**

Note that at many schools all but the Substitution Rule tend to be taught in a Calculus II class.

**u Substitution**: The substitution \( u = g(x) \) will convert \( \int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du \) using \( du = g'(x)\,dx \). For indefinite integrals drop the limits of integration.

| Ex. | \( \int_{1}^{2} 5x^2 \cos (x^3)\,dx \) | \( \int_{1}^{2} 5x^2 \cos (x^3)\,dx = \int_{\frac{1}{3}}^{\frac{8}{3}} \cos (u)\,du \)
|-----|---------------------------------|--------------------------------------------------|
|     | \( u = x^3 \Rightarrow du = 3x^2\,dx \Rightarrow x^2\,dx = \frac{1}{3}\,du \) | \( = \frac{5}{3}\,\sin (u)\bigg|_{1}^{8} = \frac{5}{3}\,(\sin (8) - \sin (1)) \)
|     | \( x = 1 \Rightarrow u = 1^3 = 1 \) \( \therefore \) \( x = 2 \Rightarrow u = 2^3 = 8 \)

**Integration by Parts**: \( \int u\,dv = uv - \int v\,du \) and \( \int_{a}^{b} u\,dv = uv \bigg|_{a}^{b} - \int_{a}^{b} v\,du \). Choose \( u \) and \( dv \) from integral and compute \( du \) by differentiating \( u \) and compute \( v \) using \( \int dv \).

| Ex. | \( \int_{1}^{5} x^2 e^{-x}\,dx \) | \( \int_{3}^{5} \ln x\,dx \)
|-----|---------------------------------|----------------------------------|
|     | \( u = x \) \( \Rightarrow \) \( dv = e^{-x}\,dx \) \( \Rightarrow \) \( du = dx \) \( \Rightarrow \) \( v = -e^{-x} \) | \( u = \ln x \) \( \Rightarrow \) \( dv = dx \) \( \Rightarrow \) \( du = \frac{1}{x}\,dx \) \( \Rightarrow \) \( v = x \)
|     | \( \int x^2 e^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + c \) | \( \int_{3}^{5} \ln x\,dx = x\ln x\bigg|_{3}^{5} - \int_{3}^{5} dx = (x\ln (x) - x)\bigg|_{3}^{5} \)
|     |                                           | \( = 5\ln (5) - 3\ln (3) - 2 \)

**Products and (some) Quotients of Trig Functions**

For \( \int \sin^n x\,\cos^m x\,dx \) we have the following:

1. **n odd**. Strip 1 sine out and convert rest to cosines using \( \sin^2 x = 1 - \cos^2 x \), then use the substitution \( u = \cos x \).
2. **m odd**. Strip 1 cosine out and convert rest to sines using \( \cos^2 x = 1 - \sin^2 x \), then use the substitution \( u = \sin x \).
3. **n and m both odd**. Use either 1. or 2.
4. **n and m both even**. Use double angle and/or half angle formulas to reduce the integral into a form that can be integrated.

**Trig Formulas**:

\( \sin (2x) = 2\sin (x)\cos (x) \), \( \cos^2 (x) = \frac{1}{2}(1 + \cos (2x)) \), \( \sin^2 (x) = \frac{1}{2}(1 - \cos (2x)) \)

| Ex. | \( \int \tan^3 x\,\sec^3 x\,dx \) | \( \int \sin^3 x\,\cos^3 x\,dx \)
|-----|---------------------------------|----------------------------------|
|     | \( \int \tan^3 x\,\sec^3 x\,dx = \int \tan^2 x\,\sec^4 x\,\tan x\,\sec x\,dx \) | \( \int \sin^3 x\,\cos^3 x\,dx = \int \sin^4 x\,\sin x\,dx = \int \left(\frac{\sin^2 x}{\cos^3 x}\right)^2\sin x\,dx \)
|     | \( = \int (\sec^2 x - 1)\sec^4 x\,\tan x\,\sec x\,dx \) | \( = \int (\frac{1 - \cos^2 x}{\cos^3 x})^2\sin x\,dx \)
|     | \( = \int (u^2 - 1)u^4\,du \) \( \Rightarrow \) \( u = \sec x \) | \( = \int \left(\frac{1 - u^2}{u^3}\right)^2 u^2\,du \) \( \Rightarrow \) \( u = \cos x \)
|     | \( = \frac{1}{2}\sec^2 x - \frac{1}{2}\sec^4 x + c \) | \( = -\int \left(\frac{1 - 2u^2 + u^4}{u^3}\right)\,du \) \( \Rightarrow \) \( u = \cos x \)
|     |                                           | \( = \frac{1}{2}\sec^2 x + 2\ln |\cos x| - \frac{1}{2}\cos^2 x + c \)

Visit [http://tutorial.math.lamar.edu](http://tutorial.math.lamar.edu) for a complete set of Calculus notes. © 2005 Paul Dawkins
### Trig Substitutions

If the integral contains the following root use the given substitution and formula to convert into an integral involving trig functions.

\[ \sqrt{a^2 - b^2x^2} \Rightarrow x = \frac{a}{b} \sin \theta \quad \text{cos}^2 \theta = 1 - \sin^2 \theta \]

\[ \sqrt{b^2x^2 - a^2} \Rightarrow x = \frac{a}{b} \sec \theta \quad \tan \theta = \sec^2 \theta - 1 \]

\[ \sqrt{a^2 + b^2x^2} \Rightarrow x = \frac{a}{b} \tan \theta \quad \sec^2 \theta = 1 + \tan^2 \theta \]

**Ex.**

\[ \int \frac{16}{x^2\sqrt{4-9x^2}} \, dx \]
\[ x = \frac{2}{3} \sin \theta \Rightarrow \quad dx = \frac{2}{3} \cos \theta \, d\theta \]
\[ \sqrt{4-9x^2} = \sqrt{4-4\sin^2 \theta} = 2\cos \theta \]

Recall \( \sqrt{x^2} = |x| \). Because we have an indefinite integral we’ll assume positive and drop absolute value bars. If we had a definite integral we’d need to compute \( \theta 's \) and remove absolute value bars based on that and,

\[ |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \]

In this case we have \( \sqrt{4-9x^2} = 2 \cos \theta \).

### Partial Fractions

If integrating \( \int P(x) \frac{dx}{Q(x)} \) where the degree of \( P(x) \) is smaller than the degree of \( Q(x) \). Factor denominator as completely as possible and find the partial fraction decomposition of the rational expression. Integrate the partial fraction decomposition (P.F.D.). For each factor in the denominator we get term(s) in the decomposition according to the following table.

<table>
<thead>
<tr>
<th>Factor in ( Q(x) )</th>
<th>Term in P.F.D</th>
<th>Factor in ( Q(x) )</th>
<th>Term in P.F.D</th>
</tr>
</thead>
<tbody>
<tr>
<td>( ax + b )</td>
<td>( \frac{A}{ax + b} )</td>
<td>( (ax + b)^k )</td>
<td>( \frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_k}{(ax + b)^k} )</td>
</tr>
<tr>
<td>( ax^2 + bx + c )</td>
<td>( \frac{Ax + B}{ax^2 + bx + c} )</td>
<td>( (ax^2 + bx + c)^k )</td>
<td>( \frac{A_1x + B_1}{ax^2 + bx + c} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k} )</td>
</tr>
</tbody>
</table>

**Ex.**

\[ \int \frac{7x^2 + 13x}{(x-1)(x^2+4)} \, dx \]
\[ \int \frac{7x^2 + 13x}{(x-1)(x^2+4)} \, dx = \int \frac{4}{x-1} + \frac{3x + 16}{x^2+4} \, dx \]
\[ = \int \frac{4}{x-1} + \frac{3x}{x^2+4} + \frac{16}{x^2+4} \, dx \]
\[ = 4 \ln |x-1| + \frac{3}{2} \ln (x^2+4) + 8 \tan^{-1} \left( \frac{x}{2} \right) \]

Here is partial fraction form and recombined.

An alternate method that *sometimes* works to find constants. Start with setting numerators equal in previous example: \( 7x^2 + 13x = A \left( x^2 + 4 \right) + Bx + C \left( x - 1 \right) \). Chose *nice* values of \( x \) and plug in.

For example if \( x = 1 \) we get \( 20 = 5A \) which gives \( A = 4 \). This won’t always work easily.
Applications of Integrals

Net Area: \( \int_a^b f(x) \, dx \) represents the net area between \( f(x) \) and the \( x \)-axis with area above \( x \)-axis positive and area below \( x \)-axis negative.

Area Between Curves: The general formulas for the two main cases for each are,

\[ y = f(x) \Rightarrow A = \int_a^b [\text{upper function}] - [\text{lower function}] \, dx \quad \text{&} \quad x = f(y) \Rightarrow A = \int_c^d [\text{right function}] - [\text{left function}] \, dy \]

If the curves intersect then the area of each portion must be found individually. Here are some sketches of a couple possible situations and formulas for a couple of possible cases.

Volumes of Revolution: The two main formulas are \( V = \int A(x) \, dx \) and \( V = \int A(y) \, dy \). Here is some general information about each method of computing and some examples.

Rings
\[ A = \pi \left( (\text{outer radius})^2 - (\text{inner radius})^2 \right) \]

Limits: \( x/y \) of right/bot ring to \( x/y \) of left/top ring
Horz. Axis use \( f(x) \), Vert. Ax. use \( f(y) \), \( g(x) \), \( A(x) \) and \( dx \).
\[ g(y) \), \( A(y) \) and \( dy \).

Cylinders
\[ A = 2\pi \text{(radius)} \left( \text{width / height} \right) \]

Limits: \( x/y \) of inner cyl. to \( x/y \) of outer cyl.
Horz. Axis use \( f(y) \), Vert. Ax. use \( f(x) \), \( g(y) \), \( A(y) \) and \( dy \).
\[ g(x) \), \( A(x) \) and \( dx \).

Ex. Axis: \( y = a > 0 \)  \hspace{1cm} Ex. Axis: \( y = a \leq 0 \)  \hspace{1cm} Ex. Axis: \( y = a > 0 \)  \hspace{1cm} Ex. Axis: \( y = a \leq 0 \)

These are only a few cases for horizontal axis of rotation. If axis of rotation is the \( x \)-axis use the \( y = a \leq 0 \) case with \( a = 0 \). For vertical axis of rotation (\( x = a > 0 \) and \( x = a \leq 0 \)) interchange \( x \) and \( y \) to get appropriate formulas.
Work : If a force of \( F(x) \) moves an object in \( a \leq x \leq b \), the work done is \( W = \int_a^b F(x) \, dx \)

Average Function Value : The average value of \( f(x) \) on \( a \leq x \leq b \) is 
\[
 f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) \, dx
\]

Arc Length Surface Area : Note that this is often a Calc II topic. The three basic formulas are,
\[
 L = \int_a^b ds 
\]
\[
 S_{A} = \int_a^b 2\pi y \, ds \quad \text{(rotate about x-axis)} 
\]
\[
 S_{A} = \int_a^b 2\pi x \, ds \quad \text{(rotate about y-axis)} 
\]
where \( ds \) is dependent upon the form of the function being worked with as follows.
\[
 ds = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \quad \text{if } y = f(x), \ a \leq x \leq b 
\]
\[
 ds = \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy \quad \text{if } x = f(y), \ a \leq y \leq b 
\]
With surface area you may have to substitute in for the \( x \) or \( y \) depending on your choice of \( ds \) to match the differential in the \( ds \). With parametric and polar you will always need to substitute.

Improper Integral
An improper integral is an integral with one or more infinite limits and/or discontinuous integrands. Integral is called convergent if the limit exists and has a finite value and divergent if the limit doesn’t exist or has infinite value. This is typically a Calc II topic.

Infinite Limit
1. \( \int_a^\infty f(x) \, dx = \lim_{t \to \infty} \int_t^a f(x) \, dx \)
2. \( \int_{-\infty}^b f(x) \, dx = \lim_{t \to -\infty} \int_t^b f(x) \, dx \)
3. \( \int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^c f(x) \, dx + \int_c^{\infty} f(x) \, dx \) provided BOTH integrals are convergent.

Discontinuous Integrand
1. Discont. at \( a \): \( \int_a^b f(x) \, dx = \lim_{t \to a^+} \int_t^b f(x) \, dx \)
2. Discont. at \( b \): \( \int_a^b f(x) \, dx = \lim_{t \to b^-} \int_a^t f(x) \, dx \)
3. Discontinuity at \( a < c < b \) : \( \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx \) provided both are convergent.

Comparison Test for Improper Integrals : If \( f(x) \geq g(x) \geq 0 \) on \( [a, \infty) \) then,
1. If \( \int_a^\infty f(x) \, dx \) conv. then \( \int_a^\infty g(x) \, dx \) conv.
2. If \( \int_a^\infty g(x) \, dx \) divg. then \( \int_a^\infty f(x) \, dx \) divg.

Useful fact : If \( a > 0 \) then \( \int_a^\infty \frac{1}{x^p} \, dx \) converges if \( p > 1 \) and diverges for \( p \leq 1 \).

Approximating Definite Integrals
For given integral \( \int_a^b f(x) \, dx \) and a \( n \) (must be even for Simpson’s Rule) define \( \Delta x = \frac{b-a}{n} \) and divide \( [a,b] \) into \( n \) subintervals \( [x_0,x_1], [x_1,x_2], \ldots, [x_{n-1},x_n] \) with \( x_0 = a \) and \( x_n = b \) then,

Midpoint Rule : \( \int_a^b f(x) \, dx \approx \Delta x \left[ f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*) \right] \), \( x_i^* \) is midpoint \( [x_{i-1},x_i] \)

Trapezoid Rule : \( \int_a^b f(x) \, dx \approx \frac{\Delta x}{2} \left[ f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n) \right] \)

Simpson’s Rule : \( \int_a^b f(x) \, dx \approx \frac{\Delta x}{3} \left[ f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right] \)