Calculus Cheat Sheet

Derivatives
Definition and Notation
If \( y = f(x) \) then the derivative is defined to be \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \).

If \( y = f(x) \) then all of the following are equivalent notations for the derivative.
\[
f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = Df(x) = \frac{dy}{dx}
\]

\( f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \) for all \( x \neq a \).

Interpretation of the Derivative
If \( y = f(x) \) then,
1. \( f'(a) \) is the instantaneous rate of change of \( f(x) \) at \( x = a \).
2. If \( f(x) \) is the position of an object at time \( x \) then \( f'(a) \) is the velocity of the object at \( x = a \).
3. \( m = f'(a) \) is the slope of the tangent line to \( y = f(x) \) at \( x = a \) and the equation of the tangent line at \( x = a \) is given by \( y = f(a) + f'(a)(x-a) \).

Basic Properties and Formulas
If \( f(x) \) and \( g(x) \) are differentiable functions (the derivative exists), \( c \) and \( n \) are any real numbers,
\[
\begin{align*}
1. \quad & (c f)' = c f'(x) \\
2. \quad & (f \pm g)' = f'(x) \pm g'(x) \\
3. \quad & (f g)' = f' g + f g' \\
4. \quad & \left( \frac{f}{g} \right)' = \frac{f' g - f g'}{g^2} \quad \text{Quotient Rule}
\end{align*}
\]

Common Derivatives
\[
\begin{align*}
\frac{d}{dx}(x) &= 1 \\
\frac{d}{dx}(\cos x) &= -\sin x \\
\frac{d}{dx}(\sin x) &= \cos x \\
\frac{d}{dx}(\tan x) &= \sec^2 x \\
\frac{d}{dx}(\csc x) &= -\csc x \cot x \\
\frac{d}{dx}(\cot x) &= -\csc^2 x \\
\frac{d}{dx}(e^x) &= e^x \\
\frac{d}{dx}(\ln x) &= \frac{1}{x} \\
\frac{d}{dx}(a^x) &= a^x \ln(a) \\
\frac{d}{dx}(\log_a(x)) &= \frac{1}{x \ln a}
\end{align*}
\]

Chain Rule Variants
The chain rule applied to some specific functions.
1. \( \frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) \)
2. \( \frac{d}{dx}[e^{f(x)}] = f'(x)e^{f(x)} \)
3. \( \frac{d}{dx}[\ln(f(x))] = \frac{f'(x)}{f(x)} \)
4. \( \frac{d}{dx}[\sin(f(x))] = f'(x)\cos(f(x)) \)
5. \( \frac{d}{dx}[\cos(f(x))] = -f'(x)\sin(f(x)) \)
6. \( \frac{d}{dx}[\tan(f(x))] = f'(x)\sec^2(f(x)) \)
7. \( \frac{d}{dx}[\sec(f(x))] = f'(x)\sec(f(x))\tan(f(x)) \)
8. \( \frac{d}{dx}[\tan^{-1}(f(x))] = \frac{f'(x)}{1 + [f(x)]^2} \)

Higher Order Derivatives
The second derivative is denoted as \( f''(x) \) and is defined as \( f''(x) = f'(x) \), i.e., the derivative of the first derivative, \( f'(x) \).

The \( n \)th derivative is denoted as \( f^{(n)}(x) \) and is defined as \( f^{(n)}(x) = f^{(n-1)}(x) \), i.e., the derivative of the \( (n-1) \)th derivative, \( f^{(n-1)}(x) \).

Implicit Differentiation
Find \( y' \) if \( e^{x+y} + x^2 + y^2 = \sin(y) + 11x \). Remember \( y = y(x) \) here, so products/quotients of \( x \) and \( y \) will use the product/quotient rule and derivatives of \( y \) will use the chain rule. The “trick” is to differentiate as normal and every time you differentiate a \( y \) you tack on a \( y' \) (from the chain rule).

After differentiating solve for \( y' \).
\[
e^{x+y}(2+2y) + 2x + 2y y' = \cos(y) y' + 11 \\
2e^{x+y} - 9y e^{x+y} + 3x^2 y^2 + 2x y y' = \cos(y) y' + 11 \\
(2x^2 y - 9e^{x+y} - \cos(y)) y' = 11 - 2e^{x+y} - 3x^2 y^2 \\
\Rightarrow y' = \frac{11 - 2e^{x+y} - 3x^2 y^2}{2x y - 9e^{x+y} - \cos(y)}
\]

Increasing/Decreasing – Concave Up/Concave Down
Critical Points
\( x = c \) is a critical point of \( f(x) \) provided either
1. \( f'(c) = 0 \) or \( 2. f'(c) \) doesn’t exist.

Increasing/Decreasing
1. If \( f'(x) > 0 \) for all \( x \) in an interval \( I \) then \( f(x) \) is increasing on the interval \( I \).
2. If \( f'(x) < 0 \) for all \( x \) in an interval \( I \) then \( f(x) \) is decreasing on the interval \( I \).

Inflection Points
\( x = c \) is a inflection point of \( f(x) \) if the concavity changes at \( x = c \).

Concave Up/Concave Down
1. If \( f''(x) > 0 \) for all \( x \) in an interval \( I \) then \( f(x) \) is concave up on the interval \( I \).
2. If \( f''(x) < 0 \) for all \( x \) in an interval \( I \) then \( f(x) \) is concave down on the interval \( I \).
### Extrema

**Absolute Extrema**
1. \( x = c \) is an absolute maximum of \( f(x) \)
   if \( f(c) \geq f(x) \) for all \( x \) in the domain.
2. \( x = c \) is an absolute minimum of \( f(x) \)
   if \( f(c) \leq f(x) \) for all \( x \) in the domain.

**Relative (local) Extrema**
1. \( x = c \) is a relative (or local) maximum of \( f(x) \)
   if \( f(c) \geq f(x) \) for all \( x \) near \( c \).
2. \( x = c \) is a relative (or local) minimum of \( f(x) \)
   if \( f(c) \leq f(x) \) for all \( x \) near \( c \).

**Fermat’s Theorem**
If \( f(x) \) has a relative (or local) extremum at \( x = c \), then \( x = c \) is a critical point of \( f(x) \).

**Extreme Value Theorem**
If \( f(x) \) is continuous on the closed interval \([a, b]\) then there exist numbers \( c \) and \( d \) so that,
1. \( a \leq c, d \leq b \), 2. \( f(c) \) is the abs. max. in \( [a, b] \), 3. \( f(d) \) is the abs. min. in \([a, b] \).

### Finding Absolute Extrema
To find the absolute extrema of the continuous function \( f(x) \) on the interval \([a, b]\) use the following process.
1. Find all critical points of \( f(x) \) in \([a, b]\).
2. Evaluate \( f(x) \) at all points found in Step 1.
3. Evaluate \( f(a) \) and \( f(b) \).
4. Identify the abs. max. (largest function value) and the abs. min. (smallest function value) from the evaluations in Steps 2 & 3.

**Mean Value Theorem**
If \( f(x) \) is continuous on the closed interval \([a, b]\) and differentiable on the open interval \((a, b)\)
then there is a number \( c \) such that \( f'(c) = \frac{f(b) - f(a)}{b - a} \).

**Newton’s Method**
If \( x_n \) is the \( n \)th guess for the root/solution of \( f(x) = 0 \) then \( (n+1) \)th guess is \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \)
provided \( f'(x) \) exists.

### Related Rates

**Ex.** A 15 foot ladder is resting against a wall.
The bottom is initially 10 ft away and is being pushed towards the wall at \( \frac{1}{2}\) ft/sec. How fast is the top moving after 12 sec?

\[
\begin{align*}
x' &= \frac{15^2 - 7^2}{\sqrt{176}} \\
&= \frac{7}{4\sqrt{176}} \\
&= 0.3112 \text{ ft/min}
\end{align*}
\]

**Optimization**

Sketch picture if needed, write down equation to be optimized and constraint. Solve for one of the two variables and plug into first equation. Find critical points of equation in range of variables and verify that they are min/max as needed.

**Ex.** We’re enclosing a rectangular field with 500 ft of fence material and one side of the field is a building. Determine dimensions that will maximize the enclosed area.

Maximize \( A = xy \) subject to constraint of \( x + 2y = 500 \). Solve constraint for \( x \) and plug into area.

\[
x = 500 - 2y \\
A = y(500 - 2y) \\
= 500y - 2y^2
\]

Differentiate and find critical point(s).

\[
A' = 500 - 4y \\
y = 125
\]

By 2nd deriv. test this is a rel. max. and so is the answer we’re after. Finally, find \( x \).

\[
x = 500 - 2(125) = 250
\]

The dimensions are then 250 x 125.

**Ex.** Determine point(s) on \( y = x^2 + 1 \) that are closest to \((0,2)\).

Minimize \( f = d^2 = (x-0)^2 + (y-2)^2 \) and the constraint is \( y = x^2 + 1 \). Solve for \( x^2 \) and plug into the function.

\[
x^2 = y - 1 \Rightarrow f = x^2 + (y-2)^2 \\
= y - 1 + (y-2)^2 = y^2 - 3y + 3
\]

Differentiate and find critical point(s).

\[
f' = 2y - 3 \\
y = \frac{3}{2}
\]

By the 2nd derivative test this is a rel. min. and so all we need to do is find \( y \) value(s).

\[
x^2 = \frac{3}{4} - 1 = \frac{1}{4} \Rightarrow x = \pm \frac{1}{2}
\]

The 2 points are then \( \left( \frac{1}{2}, \frac{3}{4} \right) \) and \( \left( -\frac{1}{2}, \frac{3}{4} \right) \).

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