# Derivatives

## Definition and Notation

If $y=f(x)$ then the derivative is defined to be $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

If $y=f(x)$ then all of the following are equivalent notations for the derivative.

$$f'(x)=y'=\frac{df}{dx}=\frac{dy}{dx}=\frac{d}{dx}(f(x))=Df(x)$$

If $y=f(x)$ all of the following are equivalent notations for derivative evaluated at $x=a$.

$$f'(a)=y'\big|_{x=a}=\frac{df}{dx}\Big|_{x=a}=\frac{dy}{dx}\Big|_{x=a}=Df(a)$$

## Interpretation of the Derivative

If $y=f(x)$ then,

1. $m=f'(a)$ is the slope of the tangent line to $y=f(x)$ at $x=a$ and the equation of the tangent line at $x=a$ is given by $y=f(a)+f'(a)(x-a)$.
2. $f'(a)$ is the instantaneous rate of change of $f(x)$ at $x=a$.
3. If $f(x)$ is the position of an object at time $x$ then $f'(a)$ is the velocity of the object at $x=a$.

## Basic Properties and Formulas

If $f(x)$ and $g(x)$ are differentiable functions (the derivative exists), $c$ and $n$ are any real numbers,

1. $(cf)'=cf'(x)$
2. $(f\pm g)'=f'(x)\pm g'(x)$
3. $(fg)'=f'g+fg'$ – **Product Rule**
4. $\left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2}$ – **Quotient Rule**
5. $\frac{d}{dx}(c)=0$
6. $\frac{d}{dx}(x^n)=nx^{n-1}$ – **Power Rule**
7. $\frac{d}{dx}(f(g(x)))=f'(g(x))g'(x)$

   This is the **Chain Rule**

## Common Derivatives

$\frac{d}{dx}(x)=1$

$\frac{d}{dx}(\sin x)=\cos x$

$\frac{d}{dx}(\cos x)=-\sin x$

$\frac{d}{dx}(\tan x)=\sec^2 x$

$\frac{d}{dx}(\sec x)=\sec x\tan x$

$\frac{d}{dx}(\csc x)=-\csc x\cot x$

$\frac{d}{dx}(\cot x)=-\csc^2 x$

$\frac{d}{dx}(\sin^{-1} x)=\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\cos^{-1} x)=-\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1} x)=\frac{1}{1+x^2}$

$\frac{d}{dx}(a^x)=a^x\ln(a)$

$\frac{d}{dx}(\mathbf{e}^x)=\mathbf{e}^x$

$\frac{d}{dx}(\ln(x))=\frac{1}{x},\quad x>0$

$\frac{d}{dx}(\ln|x|)=\frac{1}{x},\quad x\neq 0$

$\frac{d}{dx}(\log_a(x))=\frac{1}{x\ln a},\quad x>0$







## Chain Rule Variants

The chain rule applied to some specific functions.

1. $\frac{d}{dx}\left(\left[f(x)\right]^n\right)=n\left[f(x)\right]^{n-1}f'(x)$
2. $\frac{d}{dx}\left(\mathbf{e}^{f(x)}\right)=f'(x)\mathbf{e}^{f(x)}$
3. $\frac{d}{dx}\left(\ln\left[f(x)\right]\right)=\frac{f'(x)}{f(x)}$
4. $\frac{d}{dx}\left(\sin\left[f(x)\right]\right)=f'(x)\cos\left[f(x)\right]$
5. $\frac{d}{dx}\left(\cos\left[f(x)\right]\right)=-f'(x)\sin\left[f(x)\right]$
6. $\frac{d}{dx}\left(\tan\left[f(x)\right]\right)=f'(x)\sec^2\left[f(x)\right]$
7. $\frac{d}{dx}\left(\sec\left[f(x)\right]\right)=f'(x)\sec\left[f(x)\right]\tan\left[f(x)\right]$
8. $\frac{d}{dx}\left(\tan^{-1}\left[f(x)\right]\right)=\frac{f'(x)}{1+\left[f(x)\right]^2}$

## Higher Order Derivatives

The Second Derivative is denoted as

$f''(x)=f^{(2)}(x)=\frac{d^2f}{dx^2}$ and is defined as

$f''(x)=(f'(x))'$, *i.e.* the derivative of the first derivative, $f'(x)$.

The n$^{\text{th}}$ Derivative is denoted as

$f^{(n)}(x)=\frac{d^nf}{dx^n}$ and is defined as

$f^{(n)}(x)=\left(f^{(n-1)}(x)\right)'$, *i.e.* the derivative of the ($n$-1)$^{\text{st}}$ derivative, $f^{(n-1)}(x)$.

## Implicit Differentiation

Find $y'$ if $\mathbf{e}^{2x-9y}+x^3y^2=\sin(y)+11x$. Remember $y=y(x)$ here, so products/quotients of $x$ and $y$ will use the product/quotient rule and derivatives of $y$ will use the chain rule. The "trick" is to differentiate as normal and every time you differentiate a $y$ you tack on a $y'$ (from the chain rule). After differentiating solve for $y'$.

$$\mathbf{e}^{2x-9y}(2-9y')+3x^2y^2+2x^3y\,y'=\cos(y)\,y'+11$$
$$2\mathbf{e}^{2x-9y}-9y'\mathbf{e}^{2x-9y}+3x^2y^2+2x^3y\,y'=\cos(y)\,y'+11\quad\Rightarrow\quad y'=\frac{11-2\mathbf{e}^{2x-9y}-3x^2y^2}{2x^3y-9\mathbf{e}^{2x-9y}-\cos(y)}$$
$$\left(2x^3y-9\mathbf{e}^{2x-9y}-\cos(y)\right)y'=11-2\mathbf{e}^{2x-9y}-3x^2y^2$$

## Increasing/Decreasing – Concave Up/Concave Down

### Critical Points

$x=c$ is a critical point of $f(x)$ provided either **1.** $f'(c)=0$ or **2.** $f'(c)$ doesn't exist.

### Increasing/Decreasing

1. If $f'(x)>0$ for all $x$ in an interval $I$ then $f(x)$ is increasing on the interval $I$.
2. If $f'(x)<0$ for all $x$ in an interval $I$ then $f(x)$ is decreasing on the interval $I$.
3. If $f'(x)=0$ for all $x$ in an interval $I$ then $f(x)$ is constant on the interval $I$.

### Concave Up/Concave Down

1. If $f''(x)>0$ for all $x$ in an interval $I$ then $f(x)$ is concave up on the interval $I$.
2. If $f''(x)<0$ for all $x$ in an interval $I$ then $f(x)$ is concave down on the interval $I$.

### Inflection Points

$x=c$ is a inflection point of $f(x)$ if the concavity changes at $x=c$.

## Extrema

**Absolute Extrema**

1. $x = c$ is an absolute maximum of $f(x)$ if $f(c) \ge f(x)$ for all $x$ in the domain.
2. $x = c$ is an absolute minimum of $f(x)$ if $f(c) \le f(x)$ for all $x$ in the domain.

**Fermat's Theorem**

If $f(x)$ has a relative (or local) extrema at $x = c$, then $x = c$ is a critical point of $f(x)$.

**Extreme Value Theorem**

If $f(x)$ is continuous on the closed interval $[a,b]$ then there exist numbers $c$ and $d$ so that, **1.** $a \le c, d \le b$, **2.** $f(c)$ is the abs. max. in $[a,b]$, **3.** $f(d)$ is the abs. min. in $[a,b]$.

**Finding Absolute Extrema**

To find the absolute extrema of the continuous function $f(x)$ on the interval $[a,b]$ use the following process.

1. Find all critical points of $f(x)$ in $[a,b]$.
2. Evaluate $f(x)$ at all points found in Step 1.
3. Evaluate $f(a)$ and $f(b)$.
4. Identify the abs. max. (largest function value) and the abs. min.(smallest function value) from the evaluations in Steps 2 & 3.

**Relative (local) Extrema**

1. $x = c$ is a relative (or local) maximum of $f(x)$ if $f(c) \ge f(x)$ for all $x$ near $c$.
2. $x = c$ is a relative (or local) minimum of $f(x)$ if $f(c) \le f(x)$ for all $x$ near $c$.

**1st Derivative Test**

If $x = c$ is a critical point of $f(x)$ then $x = c$ is

1. a rel. max. of $f(x)$ if $f'(x) > 0$ to the left of $x = c$ and $f'(x) < 0$ to the right of $x = c$.
2. a rel. min. of $f(x)$ if $f'(x) < 0$ to the left of $x = c$ and $f'(x) > 0$ to the right of $x = c$.
3. not a relative extrema of $f(x)$ if $f'(x)$ is the same sign on both sides of $x = c$.

**2nd Derivative Test**

If $x = c$ is a critical point of $f(x)$ such that $f'(c) = 0$ then $x = c$

1. is a relative maximum of $f(x)$ if $f''(c) < 0$.
2. is a relative minimum of $f(x)$ if $f''(c) > 0$.
3. may be a relative maximum, relative minimum, or neither if $f''(c) = 0$.

**Finding Relative Extrema and/or Classify Critical Points**

1. Find all critical points of $f(x)$.
2. Use the 1st derivative test or the 2nd derivative test on each critical point.

## Mean Value Theorem

If $f(x)$ is continuous on the closed interval $[a,b]$ and differentiable on the open interval $(a,b)$ then there is a number $a < c < b$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

## Newton's Method

If $x_n$ is the $n^{th}$ guess for the root/solution of $f(x) = 0$ then $(n+1)^{st}$ guess is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ provided $f'(x_n)$ exists.

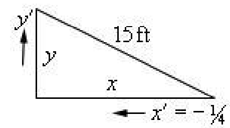




## Related Rates

Sketch picture and identify known/unknown quantities. Write down equation relating quantities and differentiate with respect to $t$ using implicit differentiation (*i.e.* add on a derivative every time you differentiate a function of $t$). Plug in known quantities and solve for the unknown quantity.

**Ex.** A 15 foot ladder is resting against a wall. The bottom is initially 10 ft away and is being pushed towards the wall at $\frac{1}{4}$ ft/sec. How fast is the top moving after 12 sec?



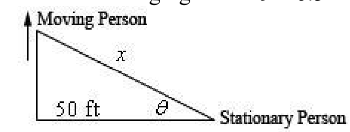

$x'$ is negative because $x$ is decreasing. Using Pythagorean Theorem and differentiating,

$$x^2 + y^2 = 15^2 \quad \Rightarrow \quad 2x\,x' + 2y\,y' = 0$$

After 12 sec we have $x = 10 - 12\left(\frac{1}{4}\right) = 7$ and so $y = \sqrt{15^2 - 7^2} = \sqrt{176}$. Plug in and solve for $y'$.

$$7\left(-\tfrac{1}{4}\right) + \sqrt{176}\; y' = 0 \;\Rightarrow\; y' = \frac{7}{4\sqrt{176}} \text{ ft/sec}$$

**Ex.** Two people are 50 ft apart when one starts walking north. The angle $\theta$ changes at 0.01 rad/min. At what rate is the distance between them changing when $\theta = 0.5$ rad?




We have $\theta' = 0.01$ rad/min. and want to find $x'$. We can use various trig fcns but easiest is,

$$\sec\theta = \frac{x}{50} \quad \Rightarrow \quad \sec\theta\tan\theta\;\theta' = \frac{x'}{50}$$

We know $\theta = 0.5$ so plug in $\theta'$ and solve.

$$\sec(0.5)\tan(0.5)(0.01) = \frac{x'}{50}$$

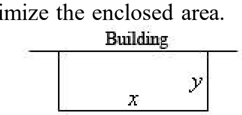
$$x' = 0.3112 \text{ ft/sec}$$

Remember to have calculator in radians!

## Optimization

Sketch picture if needed, write down equation to be optimized and constraint. Solve constraint for one of the two variables and plug into first equation. Find critical points of equation in range of variables and verify that they are min/max as needed.

**Ex.** We're enclosing a rectangular field with 500 ft of fence material and one side of the field is a building. Determine dimensions that will maximize the enclosed area.




Maximize $A = xy$ subject to constraint of $x + 2y = 500$. Solve constraint for $x$ and plug into area.

$$x = 500 - 2y \quad \Rightarrow \quad \begin{aligned} A &= y(500 - 2y) \\ &= 500y - 2y^2 \end{aligned}$$

Differentiate and find critical point(s).

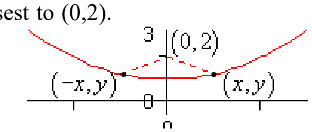
$$A' = 500 - 4y \quad \Rightarrow \quad y = 125$$

By 2nd deriv. test this is a rel. max. and so is the answer we're after. Finally, find $x$.

$$x = 500 - 2(125) = 250$$

The dimensions are then 250 x 125.

**Ex.** Determine point(s) on $y = x^2 + 1$ that are closest to (0,2).




Minimize $f = d^2 = (x - 0)^2 + (y - 2)^2$ and the constraint is $y = x^2 + 1$. Solve constraint for $x^2$ and plug into the function.

$$x^2 = y - 1 \;\Rightarrow\; f = x^2 + (y - 2)^2$$

$$= y - 1 + (y - 2)^2 = y^2 - 3y + 3$$

Differentiate and find critical point(s).

$$f' = 2y - 3 \quad \Rightarrow \quad y = \tfrac{3}{2}$$

By the 2nd derivative test this is a rel. min. and so all we need to do is find $x$ value(s).

$$x^2 = \tfrac{3}{2} - 1 = \tfrac{1}{2} \quad \Rightarrow \quad x = \pm\tfrac{1}{\sqrt{2}}$$

The 2 points are then $\left(\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$