Limits

Definitions

Precise Definition : We say $\lim_{x\to a} f(x)=L$ if for every $\varepsilon>0$ there is a $\delta>0$ such that whenever $0<|x-a|<\delta$ then $|f(x)-L|<\varepsilon$.

"Working" Definition : We say $\lim_{x\to a} f(x)=L$ if we can make $f(x)$ as close to L as we want by taking x sufficiently close to a (on either side of a) without letting $x=a$.

Right hand limit : $\lim_{x\to a^+} f(x)=L$. This has the same definition as the limit except it requires $x>a$.

Left hand limit : $\lim_{x\to a^-} f(x)=L$. This has the same definition as the limit except it requires $x<a$.

Limit at Infinity : We say $\lim_{x\to\infty} f(x)=L$ if we can make $f(x)$ as close to L as we want by taking x large enough and positive.

There is a similar definition for $\lim_{x\to-\infty} f(x)=L$ except we require x large and negative.

Infinite Limit : We say $\lim_{x\to a} f(x)=\infty$ if we can make $f(x)$ arbitrarily large (and positive) by taking x sufficiently close to a (on either side of a) without letting $x=a$.

There is a similar definition for $\lim_{x\to a} f(x)=-\infty$ except we make $f(x)$ arbitrarily large and negative.

Relationship between the limit and one-sided limits

$\lim_{x\to a} f(x)=L \Rightarrow \lim_{x\to a^+} f(x)=\lim_{x\to a^-} f(x)=L$

$\lim_{x\to a^+} f(x)=\lim_{x\to a^-} f(x)=L \Rightarrow \lim_{x\to a} f(x)=L$

$\lim_{x\to a^+} f(x)\neq\lim_{x\to a^-} f(x) \Rightarrow \lim_{x\to a} f(x)$ Does Not Exist

Properties

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and c is any number then,

1. $\lim_{x\to a}\left[cf(x)\right]=c\lim_{x\to a} f(x)$
2. $\lim_{x\to a}\left[f(x)\pm g(x)\right]=\lim_{x\to a} f(x)\pm\lim_{x\to a} g(x)$
3. $\lim_{x\to a}\left[f(x)g(x)\right]=\lim_{x\to a} f(x)\ \lim_{x\to a} g(x)$
4. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x)\neq 0$
5. $\lim_{x\to a}\left[f(x)\right]^n=\left[\lim_{x\to a} f(x)\right]^n$
6. $\lim_{x\to a}\left[\sqrt[n]{f(x)}\right]=\sqrt[n]{\lim_{x\to a} f(x)}$

Basic Limit Evaluations at $\pm\infty$

Note : $\operatorname{sgn}(a)=1$ if $a>0$ and $\operatorname{sgn}(a)=-1$ if $a<0$.

1. $\lim_{x\to\infty} \mathbf{e}^x=\infty$ & $\lim_{x\to-\infty} \mathbf{e}^x=0$
2. $\lim_{x\to\infty} \ln(x)=\infty$ & $\lim_{x\to0^+} \ln(x)=-\infty$
3. If $r>0$ then $\lim_{x\to\infty} \frac{b}{x^r}=0$
4. If $r>0$ and x^r is real for negative x then $\lim_{x\to-\infty} \frac{b}{x^r}=0$
5. n even : $\lim_{x\to\pm\infty} x^n=\infty$
6. n odd : $\lim_{x\to\infty} x^n=\infty$ & $\lim_{x\to-\infty} x^n=-\infty$
7. n even : $\lim_{x\to\pm\infty} a\,x^n+\cdots+b\,x+c=\operatorname{sgn}(a)\infty$
8. n odd : $\lim_{x\to\infty} a\,x^n+\cdots+b\,x+c=\operatorname{sgn}(a)\infty$
9. n odd : $\lim_{x\to-\infty} a\,x^n+\cdots+c\,x+d=-\operatorname{sgn}(a)\infty$

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Calculus Cheat Sheet

Evaluation Techniques

Continuous Functions

If $f(x)$ is continuous at a then $\lim_{x\to a} f(x)=f(a)$

Continuous Functions and Composition

$f(x)$ is continuous at b and $\lim_{x\to a} g(x)=b$ then

$$\lim_{x\to a} f(g(x))=f\left(\lim_{x\to a} g(x)\right)=f(b)$$

Factor and Cancel

$$\lim_{x\to2}\frac{x^2+4x-12}{x^2-2x}=\lim_{x\to2}\frac{(x-2)(x+6)}{x(x-2)}=\lim_{x\to2}\frac{x+6}{x}=\frac{8}{2}=4$$

Rationalize Numerator/Denominator

$$\lim_{x\to9}\frac{3-\sqrt{x}}{x^2-81}=\lim_{x\to9}\frac{3-\sqrt{x}}{x^2-81}\,\frac{3+\sqrt{x}}{3+\sqrt{x}}=\lim_{x\to9}\frac{9-x}{(x^2-81)(3+\sqrt{x})}=\lim_{x\to9}\frac{-1}{(x+9)(3+\sqrt{x})}=\frac{-1}{(18)(6)}=-\frac{1}{108}$$

Combine Rational Expressions

$$\lim_{h\to0}\frac{1}{h}\left(\frac{1}{x+h}-\frac{1}{x}\right)=\lim_{h\to0}\frac{1}{h}\left(\frac{x-(x+h)}{x(x+h)}\right)=\lim_{h\to0}\frac{1}{h}\left(\frac{-h}{x(x+h)}\right)=\lim_{h\to0}\frac{-1}{x(x+h)}=-\frac{1}{x^2}$$

L'Hospital's Rule

If $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$ or $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\pm\infty}{\pm\infty}$ then,

$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$ a is a number, ∞ or $-\infty$

Polynomials at Infinity

$p(x)$ and $q(x)$ are polynomials. To compute $\lim_{x\to\pm\infty}\frac{p(x)}{q(x)}$ factor largest power of x in $q(x)$ out of both $p(x)$ and $q(x)$ then compute limit.

$$\lim_{x\to-\infty}\frac{3x^2-4}{5x-2x^2}=\lim_{x\to-\infty}\frac{x^2\left(3-\frac{4}{x^2}\right)}{x^2\left(\frac{5}{x}-2\right)}=\lim_{x\to-\infty}\frac{3-\frac{4}{x^2}}{\frac{5}{x}-2}=-\frac{3}{2}$$

Piecewise Function

$\lim_{x\to-2} g(x)$ where $g(x)=\begin{cases}x^2+5 & \text{if } x<-2\\ 1-3x & \text{if } x\geq-2\end{cases}$

Compute two one sided limits,

$\lim_{x\to-2^-} g(x)=\lim_{x\to-2^-} x^2+5=9$

$\lim_{x\to-2^+} g(x)=\lim_{x\to-2^+} 1-3x=7$

One sided limits are different so $\lim_{x\to-2} g(x)$ doesn't exist. If the two one sided limits had been equal then $\lim_{x\to-2} g(x)$ would have existed and had the same value.

Some Continuous Functions

Partial list of continuous functions and the values of x for which they are continuous.

1. Polynomials for all x.
2. Rational function, except for x's that give division by zero.
3. $\sqrt[n]{x}$ (n odd) for all x.
4. $\sqrt[n]{x}$ (n even) for all $x\geq0$.
5. $\mathbf{e}^x$ for all x.
6. $\ln x$ for $x>0$.
7. $\cos(x)$ and $\sin(x)$ for all x.
8. $\tan(x)$ and $\sec(x)$ provided $x\neq\cdots,-\frac{3\pi}{2},-\frac{\pi}{2},\frac{\pi}{2},\frac{3\pi}{2},\cdots$
9. $\cot(x)$ and $\csc(x)$ provided $x\neq\cdots,-2\pi,-\pi,0,\pi,2\pi,\cdots$

Intermediate Value Theorem

Suppose that $f(x)$ is continuous on $[a, b]$ and let M be any number between $f(a)$ and $f(b)$. Then there exists a number c such that $a<c<b$ and $f(c)=M$.

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