Section 6-6 : Divergence Theorem

In this section we are going to relate surface integrals to triple integrals. We will do this with the Divergence Theorem.

**Divergence Theorem**

Let \( E \) be a simple solid region and \( S \) is the boundary surface of \( E \) with positive orientation. Let \( \vec{F} \) be a vector field whose components have continuous first order partial derivatives. Then,

\[
\int_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div} \vec{F} \, dV
\]

Let’s see an example of how to use this theorem.

**Example 1** Use the divergence theorem to evaluate \( \int_S \vec{F} \cdot d\vec{S} \) where \( \vec{F} = xy \hat{i} - \frac{1}{2} y^2 \hat{j} + z \hat{k} \) and the surface consists of the three surfaces, \( z = 4 - 3x^2 - 3y^2 \), \( 1 \leq z \leq 4 \) on the top, \( x^2 + y^2 = 1 \), \( 0 \leq z \leq 1 \) on the sides and \( z = 0 \) on the bottom.

**Solution**

Let’s start this off with a sketch of the surface.

The region \( E \) for the triple integral is then the region enclosed by these surfaces. Note that cylindrical coordinates would be a perfect coordinate system for this region. If we do that here are the limits for the ranges.

\[
0 \leq z \leq 4 - 3r^2 \\
0 \leq r \leq 1 \\
0 \leq \theta \leq 2\pi
\]
We’ll also need the divergence of the vector field so let’s get that.

\[
\text{div } \mathbf{F} = y - y + 1 = 1
\]

The integral is then,

\[
\iint_S \mathbf{F} \cdot d\mathbf{S} = 
\iiint_E \text{div } \mathbf{F} \, dV
\]

\[
= \int_0^{2\pi} \int_0^1 \int_0^{4-3r^2} r \, dz \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^1 4r - 3r^3 \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \left[ 2r^2 - \frac{3}{4}r^4 \right]_0^1 \, d\theta
\]

\[
= \int_0^{2\pi} \frac{5}{4} \, d\theta
\]

\[
= \frac{5}{2} \pi
\]