Algebra

Solutions to Practice Problems

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Preface

First, here's a little bit of history on how this material was created (there's a reason for this, I promise). A long time ago (2002 or so) when I decided I wanted to put some mathematics stuff on the web I wanted a format for the source documents that could produce both a pdf version as well as a web version of the material. After some investigation I decided to use MS Word and MathType as the easiest/quickest method for doing that. The result was a pretty ugly HTML (*i.e* web page code) and had the drawback of the mathematics were images which made editing the mathematics painful. However, it was the quickest way or dealing with this stuff.

Fast forward a few years (don't recall how many at this point) and the web had matured enough that it was now much easier to write mathematics in LaTeX (https://en.wikipedia.org/wiki/LaTeX) and have it display on the web (LaTeX was my first choice for writing the source documents). So, I found a tool that could convert the MS Word mathematics in the source documents to LaTeX. It wasn't perfect and I had to write some custom rules to help with the conversion but it was able to do it without "messing" with the mathematics and so I didn't need to worry about any math errors being introduced in the conversion process. The only problem with the tool is that all it could do was convert the mathematics and not the rest of the source document into LaTeX. That meant I just converted the math into LaTeX for the website but didn't convert the source documents.

Now, here's the reason for this history lesson. Fast forward even more years and I decided that I really needed to convert the source documents into LaTeX as that would just make my life easier and I'd be able to enable working links in the pdf as well as a simple way of producing an index for the material. The only issue is that the original tool I'd use to convert the MS Word mathematics had become, shall we say, unreliable and so that was no longer an option and it still has the problem on not converting anything else into proper LaTeX code.

So, the best option that I had available to me is to take the web pages, which already had the mathematics in proper LaTeX format, and convert the rest of the HTML into LaTeX code. I wrote a set of tools to do his and, for the most part, did a pretty decent job. The only problem is that the tools weren't perfect. So, if you run into some "odd" stuff here (things like `<sup>`, `<span>`, `</span>`, `<div>`, *etc*.) please let me know the section with the code that I missed. I did my best to find all the "orphaned" HTML code but I'm certain I missed some on occasion as I did find my eyes glazing over every once in a while as I went over the converted document.

Now, with that out of the way, here are the solutions to the practice problems for the Calculus notes.

Note that some sections will have more problems than others and some will have more or less of a variety of problems. Most sections should have a range of difficulty levels in the problems although this will vary from section to section.

Please understand as well that this document was written up with the intent of eventual display on the web. That means that some of the formatting may seem a little out of place. For example, some of the problems/parts have hints for them. Online the solution that the hint refers to will be hidden until the reader clicks on a “show” link but here that is not possible and so the hint may not be as useful here as it would be if reading this online.

Outline

Preliminaries - The purpose of this chapter is to review several topics that will arise time and again throughout this material. Many of the topics here are so important to an Algebra class that if you don't have a good working grasp of them you will find it very difficult to successfully complete the course. Also, it is assumed that you've seen the topics in this chapter somewhere prior to this class and so this chapter should be mostly a review for you. However, since most of these topics are so important to an Algebra class we will make sure that you do understand them by doing a quick review of them here.

Exponents and polynomials are integral parts of any Algebra class. If you do not remember the basic exponent rules and how to work with polynomials you will find it very difficult, if not impossible, to pass an Algebra class. This is especially true with factoring polynomials. There are more than a few sections in an Algebra course where the ability to factor is absolutely essential to being able to do the work in those sections. In fact, in many of these sections factoring will be the first step taken.

It is important that you leave this chapter with a good understanding of this material! If you don't understand this material you will find it difficult to get through the remaining chapters. Here is a brief listing of the material covered in this chapter.

Integer Exponents - In this section we will start looking at exponents. We will give the basic properties of exponents and illustrate some of the common mistakes students make in working with exponents. Examples in this section we will be restricted to integer exponents. Rational exponents will be discussed in the next section.

Rational Exponents - In this section we will define what we mean by a rational exponent and extend the properties from the previous section to rational exponents. We will also discuss how to evaluate numbers raised to a rational exponent.

Radicals - In this section we will define radical notation and relate radicals to rational exponents. We will also give the properties of radicals and some of the common mistakes students often make with radicals. We will also define simplified radical form and show how to rationalize the denominator.

Polynomials - In this section we will introduce the basics of polynomials a topic that will appear throughout this course. We will define the degree of a polynomial and discuss how to add, subtract and multiply polynomials.

Factoring Polynomials - In this section we look at factoring polynomials a topic that will appear in pretty much every chapter in this course and so is vital that you understand it. We will discuss factoring out the greatest common factor, factoring by grouping, factoring quadratics and factoring polynomials with degree greater than 2.

Rational Expressions - In this section we will define rational expressions. We will discuss how to reduce a rational expression lowest terms and how to add, subtract, multiply and divide rational expressions.

Complex Numbers - In this section we give a very quick primer on complex numbers including standard form, adding, subtracting, multiplying and dividing them.

Solving Equations and Inequalities In this chapter we will look at one of the standard topics in any Algebra class. The ability to solve equations and/or inequalities is very important and is used time and again both in this class and in later classes. We will cover a wide variety of solving topics in this chapter that should cover most of the basic equations/inequalities/techniques that are involved in solving.

Solutions and Solution Sets - In this section we introduce some of the basic notation and ideas involved in solving equations and inequalities. We define solutions for equations and inequalities and solution sets.

Linear Equations - In this section we give a process for solving linear equations, including equations with rational expressions, and we illustrate the process with several examples. In addition, we discuss a subtlety involved in solving equations that students often overlook.

Applications of Linear Equations - In this section we discuss a process for solving applications in general although we will focus only on linear equations here. We will work applications in pricing, distance/rate problems, work rate problems and mixing problems.

Equations With More Than One Variable - In this section we will look at solving equations with more than one variable in them. These equations will have multiple variables in them and we will be asked to solve the equation for one of the variables. This is something that we will be asked to do on a fairly regular basis.

Quadratic Equations, Part I - In this section we will start looking at solving quadratic equations. Specifically, we will concentrate on solving quadratic equations by factoring and the square root property in this section.

Quadratic Equations, Part II - In this section we will continue solving quadratic equations. We will use completing the square to solve quadratic equations in this section and use that to derive the quadratic formula. The quadratic formula is a quick way that will allow us to quickly solve any quadratic equation.

Quadratic Equations : A Summary - In this section we will summarize the topics from the last two sections. We will give a procedure for determining which method to use in solving quadratic equations and we will define the discriminant which will allow us to quickly determine what kind of solutions we will get from solving a quadratic equation.

Applications of Quadratic Equations - In this section we will revisit some of the applications we saw in the linear application section, only this time they will involve solving a quadratic equation. Included are examples in distance/rate problems and work rate problems.

Equations Reducible to Quadratic Form - Not all equations are in what we generally consider quadratic equations. However, some equations, with a proper substitution can be turned into a quadratic equation. These types of equations are called quadratic in form. In this section we will solve this type of equation.

Equations with Radicals - In this section we will discuss how to solve equations with square roots in them. As we will see we will need to be very careful with the potential solutions we get as the process used in solving these equations can lead to values that are not, in fact, solutions to the equation.

Linear Inequalities - In this section we will start solving inequalities. We will concentrate on solving linear inequalities in this section (both single and double inequalities). We will also introduce interval notation.

Polynomial Inequalities - In this section we will continue solving inequalities. However, in this section we move away from linear inequalities and move on to solving inequalities that involve polynomials of degree at least 2.

Rational Inequalities - We continue solving inequalities in this section. We now will solve inequalities that involve rational expressions, although as we'll see the process here is pretty much identical to the process used when solving inequalities with polynomials.

Absolute Value Equations - In this section we will give a geometric as well as a mathematical definition of absolute value. We will then proceed to solve equations that involve an absolute value. We will also work an example that involved two absolute values.

Absolute Value Inequalities - In this final section of the Solving chapter we will solve inequalities that involve absolute value. As we will see the process for solving inequalities with a $<$ (i.e. a less than) is very different from solving an inequality with a $>$ (i.e. greater than).

Graphing and Functions In this chapter we will be introducing two topics that are very important in an algebra class. We will start off the chapter with a brief discussion of graphing. This is not really the main topic of this chapter, but we need the basics down before moving into the second topic of this chapter. The next chapter will contain the remainder of the graphing discussion.

The second topic that we'll be looking at is that of functions. This is probably one of the more important ideas that will come out of an Algebra class. When first studying the concept of functions many students don't really understand the importance or usefulness of functions and function notation. The importance and/or usefulness of functions and function notation will only become apparent in later chapters and later classes. In fact, there are some topics that can only be done easily with function and function notation.

Graphing - In this section we will introduce the Cartesian (or Rectangular) coordinate system. We will define/introduce ordered pairs, coordinates, quadrants, and x and y-intercepts. We will illustrate these concepts with a quick example.

Lines - In this section we will discuss graphing lines. We will introduce the concept of slope and discuss how to find it from two points on the line. In addition, we will introduce the standard form of the line as well as the point-slope form and slope-intercept form of the line. We will finish off the section with a discussion on parallel and perpendicular lines.

Circles - In this section we discuss graphing circles. We introduce the standard form of the circle and show how to use completing the square to put an equation of a circle into standard form.

The Definition of a Function - In this section we will formally define relations and functions. We also give a “working definition” of a function to help understand just what a function is. We introduce function notation and work several examples illustrating how it works. We also define the domain and range of a function. In addition, we introduce piecewise functions in this section.

Graphing Functions - In this section we discuss graphing functions including several examples of graphing piecewise functions.

Combining functions - In this section we will discuss how to add, subtract, multiply and divide functions. In addition, we introduce the concept of function composition.

Inverse Functions - In this section we define one-to-one and inverse functions. We also discuss a process we can use to find an inverse function and verify that the function we get from this process is, in fact, an inverse function.

Common Graphs We started the process of graphing in the previous chapter. However, since the main focus of that chapter was functions we didn’t graph all that many equations or functions. In this chapter we will now look at graphing a wide variety of equations and functions.

Lines, Circles and Piecewise Functions - This section is here only to acknowledge that we’ve already talked about graphing these in a previous chapter.

Parabolas - In this section we will be graphing parabolas. We introduce the vertex and axis of symmetry for a parabola and give a process for graphing parabolas. We also illustrate how to use completing the square to put the parabola into the form $f(x) = a(x-h)^2 + k$.

Ellipses - In this section we will graph ellipses. We introduce the standard form of an ellipse and how to use it to quickly graph an ellipse.

Hyperbolas - In this section we will graph hyperbolas. We introduce the standard form of a hyperbola and how to use it to quickly graph a hyperbola.

Miscellaneous Functions - In this section we will graph a couple of common functions that don’t really take all that much work to do but will be needed in later sections. We’ll be looking at the constant function, square root, absolute value and a simple cubic function.

Transformations - In this section we will be looking at vertical and horizontal shifts of graphs as well as reflections of graphs about the x and y-axis. Collectively these are often called transformations and if we understand them they can often be used to allow us to quickly graph some fairly complicated functions.

Symmetry - In this section we introduce the idea of symmetry. We discuss symmetry about the x-axis, y-axis and the origin and we give methods for determining what, if any symmetry, a graph will have without having to actually graph the function.

Rational Functions - In this section we will discuss a process for graphing rational functions. We will also introduce the ideas of vertical and horizontal asymptotes as well as how to determine if the graph of a rational function will have them.

Polynomial Functions In this chapter we are going to take a more in depth look at polynomials. We've already solved and graphed second degree polynomials (*i.e.* quadratic equations/functions) and we now want to extend things out to more general polynomials. We will take a look at finding solutions to higher degree polynomials and how to get a rough sketch for a higher degree polynomial.

We will also be looking at Partial Fractions in this chapter. It doesn't really have anything to do with graphing polynomials but needed to be put somewhere and this chapter seemed like as good a place as any.

Dividing Polynomials - In this section we'll review some of the basics of dividing polynomials. We will define the remainder and divisor used in the division process and introduce the idea of synthetic division. We will also give the Division Algorithm.

Zeroes/Roots of Polynomials - In this section we'll define the zero or root of a polynomial and whether or not it is a simple root or has multiplicity k. We will also give the Fundamental Theorem of Algebra and The Factor Theorem as well as a couple of other useful Facts.

Graphing Polynomials - In this section we will give a process that will allow us to get a rough sketch of the graph of some polynomials. We discuss how to determine the behavior of the graph at x-intercepts and the leading coefficient test to determine the behavior of the graph as we allow x to increase and decrease without bound.

Finding Zeroes of Polynomials - As we saw in the previous section in order to sketch the graph of a polynomial we need to know what it's zeroes are. However, if we are not able to factor the polynomial we are unable to do that process. So, in this section we'll look at a process using the Rational Root Theorem that will allow us to find some of the zeroes of a polynomial and in special cases all of the zeroes.

Partial Fractions - In this section we will take a look at the process of partial fractions and finding the partial fraction decomposition of a rational expression. What we will be asking here is what "smaller" rational expressions did we add and/or subtract to get the given rational expression. This is a process that has a lot of uses in some later math classes. It can show up in Calculus and Differential Equations for example.

Exponential Functions - In this section we will introduce exponential functions. We will give some of the basic properties and graphs of exponential functions. We will also discuss what many people consider to be the exponential function, $f(x) = \mathbf{e}^x$.

Logarithm Functions - In this section we will introduce logarithm functions. We give the basic properties and graphs of logarithm functions. In addition, we discuss how to evaluate some basic logarithms including the use of the change of base formula. We will also discuss the common logarithm, log(x), and the natural logarithm, ln(x).

Solving Exponential Equations - In this section we will discuss a couple of methods for solving equations that contain exponentials.

Solving Logarithm Equations - In this section we will discuss a couple of methods for solving equations that contain logarithms. Also, as we'll see, with one of the methods we will need to be careful of the results of the method as it is always possible that the method gives values that are, in fact, not solutions to the equation.

Applications - In this section we will look at a couple of applications of exponential functions and an application of logarithms. We look at compound interest, exponential growth and decay and earthquake intensity.

Systems of Equations This is a fairly short chapter devoted to solving systems of equations. A system of equations is a set of equations each containing one or more variable.

We will focus exclusively on systems of two equations with two unknowns and three equations with three unknowns although the methods looked at here can be easily extended to more equations. Also, with the exception of the last section we will be dealing only with systems of linear equations.

Linear Systems with Two Variables - In this section we will solve systems of two equations and two variables. We will use the method of substitution and method of elimination to solve the systems in this section. We will also introduce the concepts of inconsistent systems of equations and dependent systems of equations.

Linear Systems with Three Variables - In this section we will work a couple of quick examples illustrating how to use the method of substitution and method of elimination introduced in the previous section as they apply to systems of three equations.

Augmented Matrices - In this section we will look at another method for solving systems. We will introduce the concept of an augmented matrix. This will allow us to use the method of Gauss-Jordan elimination to solve systems of equations. We will use the method with systems of two equations and systems of three equations.

More on the Augmented Matrix - In this section we will revisit the cases of inconsistent and dependent solutions to systems and how to identify them using the augmented matrix method.

Nonlinear Systems - In this section we will take a quick look at solving nonlinear systems of equations. A nonlinear system of equations is a system in which at least one of the equations is not linear, i.e. has degree of two or more. Note as well that the discussion here does not cover all the possible solution methods for nonlinear systems. Solving nonlinear systems is often a much more involved process than solving linear systems.

1 Preliminaries

This chapter consists of some material that many students in a College Algebra course should have seen somewhere prior to the course. However, these topics are so important to a College Algebra course that we need to make sure they are covered so those that haven't seen them prior to the course can get caught up and to allows those that have seen them a chance for a refresher. Most of these topics will arise time and again as we cover the rest of the material for the course and if you don't have a good grasp of them you will find it difficult to successfully complete the course.

Exponents and polynomials are integral parts of any College Algebra class. If you do not remember the basic exponent rules and how to work with polynomials you will find it very difficult, if not impossible, to pass a College Algebra class. This is especially true with factoring polynomials. There are more than a few sections in a College Algebra course where the ability to factor is absolutely essential to being able to do the work in those sections. In fact, in many of these sections factoring will be the first step taken.

So, once again, it is important that you leave this chapter with a good understanding of this material! If you don't understand this material you will find it difficult to get through the remaining chapters.

The following sections are the practice problems, with solutions, for this material.

1.1 Integer Exponents

1. Evaluate the following expression and write the answer as a single number without exponents.

$$-6^2 + 4 \cdot 3^2$$

Solution

There is not really a whole lot to this problem. All we need to do is the evaluations recalling the proper order of operations.

$$-6^2 + 4 \cdot 3^2 = -36 + 4 \cdot (9) = -36 + 36 = \boxed{0}$$

Be careful with the first term and recall that,

$$-6^2 = -\left(6^2\right) = -(36) = -36$$

If we'd wanted the minus sign to also get squared we'd have written,

$$(-6)^2 = 36$$

Always remember to be careful with exponents. The only thing that gets the exponent is the number/term immediately to the left of the exponent. If we want to include minus signs on numbers with exponents then we need to add in parenthesis.

2. Evaluate the following expression and write the answer as a single number without exponents.

$$\frac{(-2)^4}{(3^2 + 2^2)^2}$$

Solution

There is not really a whole lot to this problem. All we need to do is the evaluations recalling the proper order of operations.

$$\frac{(-2)^4}{(3^2 + 2^2)^2} = \frac{16}{(9 + 4)^2} = \frac{16}{(13)^2} = \boxed{\frac{16}{169}}$$

Remember that we need to do the evaluations inside the parenthesis in the denominator before we deal with the overall exponent that is on the parenthesis.

3. Evaluate the following expression and write the answer as a single number without exponents.

$$\frac{4^0 \cdot 2^{-2}}{3^{-1} \cdot 4^{-2}}$$

Solution

There is not really a whole lot to this problem. All we need to do is the evaluations recalling the proper order of operations.

$$\frac{4^0 \cdot 2^{-2}}{3^{-1} \cdot 4^{-2}} = \frac{4^0 \cdot 3^1 \cdot 4^2}{2^2} = \frac{(1) \cdot (3) \cdot (16)}{4} = \boxed{12}$$

It is almost always going to be best to first get rid of negative exponents prior to doing any of the rest of the evaluation work. Also, don't forget to reduce any resultant fractions down as much as possible.

4. Evaluate the following expression and write the answer as a single number without exponents.

$$2^{-1} + 4^{-1}$$

Solution

There is not really a whole lot to this problem. All we need to do is the evaluations recalling the proper order of operations.

$$2^{-1} + 4^{-1} = \frac{1}{2} + \frac{1}{4} = \boxed{\frac{3}{4}}$$

It is almost always going to be best to first get rid of negative exponents prior to doing any of the rest of the evaluation work. Also, make sure you can add/subtract fractions! We're going to be running into a lot of fractions here and you need to be able to work with those.

5. Simplify the following expression and write the answer with only positive exponents.

$$\left(2w^4v^{-5}\right)^{-2}$$

Solution

There is not really a whole lot to this problem. All we need to do is use the properties from this section to do the simplification.

$$\left(2w^4v^{-5}\right)^{-2} = 2^{-2}w^{-8}v^{10} = \frac{v^{10}}{2^2w^8} = \boxed{\frac{v^{10}}{4w^8}}$$

Note that there are several "paths" (*i.e.* the order in which you chose to use the properties) you can take to do the simplification. Each will end up with the same answer however and so you don't need to get excited if you chose a different order in which to use the properties than we did here.

6. Simplify the following expression and write the answer with only positive exponents.

$$\frac{2x^4y^{-1}}{x^{-6}y^3}$$

Solution

There is not really a whole lot to this problem. All we need to do is use the properties from this section to do the simplification.

$$\frac{2x^4y^{-1}}{x^{-6}y^3} = \frac{2x^4x^6}{y^3y^1} = \boxed{\frac{2x^{10}}{y^4}}$$

Note that there are several "paths" (*i.e.* the order in which you chose to use the properties) you can take to do the simplification. Each will end up with the same answer however and so you don't need to get excited if you chose a different order in which to use the properties than we did here.

7. Simplify the following expression and write the answer with only positive exponents.

$$\frac{m^{-2}n^{-10}}{m^{-7}n^{-3}}$$

Solution

There is not really a whole lot to this problem. All we need to do is use the properties from this section to do the simplification.

$$\frac{m^{-2}n^{-10}}{m^{-7}n^{-3}} = \frac{m^{7}n^{3}}{m^{2}n^{10}} = \boxed{\frac{m^{5}}{n^{7}}}$$

8. Simplify the following expression and write the answer with only positive exponents.

$$\frac{\left(2p^{2}\right)^{-3}q^{4}}{\left(6q\right)^{-1}p^{-7}}$$

Solution

There is not really a whole lot to this problem. All we need to do is use the properties from this section to do the simplification.

$$\frac{\left(2p^{2}\right)^{-3}q^{4}}{\left(6q\right)^{-1}p^{-7}} = \frac{2^{-3}p^{-6}q^{4}}{6^{-1}q^{-1}p^{-7}} = \frac{6^{1}p^{7}q^{4}q^{1}}{2^{3}p^{6}} = \frac{6p^{1}q^{5}}{8} = \boxed{\frac{3pq^{5}}{4}}$$

Don't try to do use too many properties all at once. Sometimes it is very easy to use too many properties all in one step and make a mistake. There's nothing wrong with using only a single property or two with each step.

9. Simplify the following expression and write the answer with only positive exponents.

$$\left(\frac{z^2 y^{-1} x^{-3}}{x^{-8} z^6 y^4}\right)^{-4}$$

Solution

There is not really a whole lot to this problem. All we need to do is use the properties from this section to do the simplification.

$$\left(\frac{z^2 y^{-1} x^{-3}}{x^{-8} z^6 y^4}\right)^{-4} = \left(\frac{z^2 x^8}{x^3 z^6 y^1 y^4}\right)^{-4} = \left(\frac{x^5}{z^4 y^5}\right)^{-4} = \left(\frac{z^4 y^5}{x^5}\right)^{4} = \boxed{\frac{z^{16} y^{20}}{x^{20}}}$$

In this case since there was a fair amount of simplification that could be done on the fraction inside the parenthesis so we decided to do that simplification prior to dealing with the exponent on the parenthesis.

1.2 Rational Exponents

1. Evaluate the following expression and write the answer as a single number without exponents.

$$36^{\frac{1}{2}}$$

Hint

Recall that $b^{\frac{1}{n}}$ is really asking what number did we raise to the n to get b. Or in other words,

$$b^{\frac{1}{n}} = ? \qquad \text{is equivalent to} \qquad ?^n = b$$

Solution

For this problem we know that $6^2 = 36$ and so we also know that,

$$36^{\frac{1}{2}} = \boxed{6}$$

Note that if you aren't sure of the answer to these kinds of problems all you really need to do is set up

$$?^2 = 36$$

and start trying integers until you get the one you need.

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2. Evaluate the following expression and write the answer as a single number without exponents.

$$(-125)^{\frac{1}{3}}$$

Hint

Recall that $b^{\frac{1}{n}}$ is really asking what number did we raise to the n to get b. Or in other words,

$$b^{\frac{1}{n}} = ? \qquad \text{is equivalent to} \qquad ?^n = b$$

Solution

For this problem we know that $5^3 = 125$. Therefore, we also know that $(-5)^3 = -125$ and so we further know that,

$$(-125)^{\frac{1}{3}} = \boxed{-5}$$

Note that if you aren't sure of the answer to these kinds of problems all you really need to do is set up

$$?^3 = -125$$

and start trying integers until you get the one you need. We also know that because the result is a negative number we had to have a negative number to start off with since we can't turn a positive number into a negative number simply by raising it to an integer.

3. Evaluate the following expression and write the answer as a single number without exponents.

$$-16^{\frac{3}{2}}$$

Hint

Don't forget your basic exponent rules and how the first two practice problems worked. Also, be careful with minus signs in this problem.

Step 1

First, let's write the problem as,

$$-\left(16^{\frac{3}{2}}\right)$$

so we aren't tempted to bring the minus sign into the exponent.

Now, let's recall our basic exponent rules and note that we can easily write this as,

$$-\left(16^{\frac{3}{2}}\right) = -\left(\left(16^{\frac{1}{2}}\right)^3\right)$$

Step 2

Now, recalling how the first two practice problems worked we can see that,

$$16^{\frac{1}{2}} = 4$$

because $4^2 = 16$.

Therefore,

$$-16^{\frac{3}{2}} = -\left(16^{\frac{3}{2}}\right) = -\left(\left(16^{\frac{1}{2}}\right)^3\right) = -\left((4)^3\right) = -(64) = \boxed{-64}$$

Sometimes the easiest way to do these kinds of problems when you first run into them is to break them up into manageable steps as we did here.

4. Evaluate the following expression and write the answer as a single number without exponents.

$$27^{-\frac{5}{3}}$$

Hint

Don't forget your basic exponent rules and how the first two practice problems worked.

Step 1

Let's first recall our basic exponent rules and note that we can easily write this as,

$$27^{-\frac{5}{3}} = \frac{1}{27^{\frac{5}{3}}} = \frac{1}{\left(27^{\frac{1}{3}}\right)^5}$$

Step 2

Now, recalling how the first two practice problems worked we can see that,

$$27^{\frac{1}{3}} = 3$$

because $3^3 = 27$.

Therefore,

$$27^{-\frac{5}{3}} = \frac{1}{27^{\frac{5}{3}}} = \frac{1}{\left(27^{\frac{1}{3}}\right)^5} = \frac{1}{(3)^5} = \boxed{\frac{1}{243}}$$

Sometimes the easiest way to do these kinds of problems when you first run into them is to break them up into manageable steps as we did here.

5. Evaluate the following expression and write the answer as a single number without exponents.

$$\left(\frac{9}{4}\right)^{\frac{1}{2}}$$

Hint

Don't forget your basic exponent rules and how the first two practice problems worked.

Step 1

Let's first recall our basic exponent rules and note that we can easily write this as,

$$\left(\frac{9}{4}\right)^{\frac{1}{2}} = \frac{9^{\frac{1}{2}}}{4^{\frac{1}{2}}}$$

Step 2

Now, recalling how the first two practice problems worked we can see that,

$$9^{\frac{1}{2}} = 3 \qquad\qquad 4^{\frac{1}{2}} = 2$$

Therefore,

$$\left(\frac{9}{4}\right)^{\frac{1}{2}} = \frac{9^{\frac{1}{2}}}{4^{\frac{1}{2}}} = \boxed{\frac{3}{2}}$$

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6. Evaluate the following expression and write the answer as a single number without exponents.

$$\left(\frac{8}{343}\right)^{-\frac{2}{3}}$$

Hint

Don't forget your basic exponent rules and how the first two practice problems worked.

Step 1

Let's first recall our basic exponent rules and note that we can easily write this as,

$$\left(\frac{8}{343}\right)^{-\frac{2}{3}} = \left(\frac{343}{8}\right)^{\frac{2}{3}} = \frac{343^{\frac{2}{3}}}{8^{\frac{2}{3}}} = \frac{\left(343^{\frac{1}{3}}\right)^2}{\left(8^{\frac{1}{3}}\right)^2}$$

Step 2

Now, recalling how the first two practice problems worked we can see that,

$$343^{\frac{1}{3}} = 7 \qquad\qquad 8^{\frac{1}{3}} = 2$$

Therefore,

$$\left(\frac{8}{343}\right)^{-\frac{2}{3}} = \left(\frac{343}{8}\right)^{\frac{2}{3}} = \frac{343^{\frac{2}{3}}}{8^{\frac{2}{3}}} = \frac{\left(343^{\frac{1}{3}}\right)^2}{\left(8^{\frac{1}{3}}\right)^2} = \frac{7^2}{2^2} = \boxed{\frac{49}{4}}$$

7. Simplify the following expression and write the answer with only positive exponents.

$$\left(a^3 b^{-\frac{1}{4}}\right)^{\frac{2}{3}}$$

Solution

There isn't really a lot to do here other than to use the exponent properties from the previous section to do the simplification.

$$\left(a^3 b^{-\frac{1}{4}}\right)^{\frac{2}{3}} = a^2 b^{-\frac{1}{6}} = \boxed{\frac{a^2}{b^{\frac{1}{6}}}}$$

8. Simplify the following expression and write the answer with only positive exponents.

$$x^{\frac{1}{4}} x^{-\frac{1}{5}}$$

Solution

There isn't really a lot to do here other than to use the exponent properties from the previous section to do the simplification.

$$x^{\frac{1}{4}} x^{-\frac{1}{5}} = x^{\frac{1}{4}-\frac{1}{5}} = \boxed{x^{\frac{1}{20}}}$$

9. Simplify the following expression and write the answer with only positive exponents.

$$\left(\frac{q^3\,p^{-\frac{1}{2}}}{q^{-\frac{1}{3}}\,p}\right)^{\frac{3}{7}}$$

Solution

There isn't really a lot to do here other than to use the exponent properties from the previous section to do the simplification.

$$\left(\frac{q^3\,p^{-\frac{1}{2}}}{q^{-\frac{1}{3}}\,p}\right)^{\frac{3}{7}} = \left(\frac{q^3\,q^{\frac{1}{3}}}{pp^{\frac{1}{2}}}\right)^{\frac{3}{7}} = \left(\frac{q^{\frac{10}{3}}}{p^{\frac{3}{2}}}\right)^{\frac{3}{7}} = \boxed{\frac{q^{\frac{10}{7}}}{p^{\frac{9}{14}}}}$$

10. Simplify the following expression and write the answer with only positive exponents.

$$\left(\frac{m^{\frac{1}{2}}\,n^{-\frac{1}{3}}}{n^{\frac{2}{3}}\,m^{-\frac{7}{4}}}\right)^{-\frac{1}{6}}$$

Solution

There isn't really a lot to do here other than to use the exponent properties from the previous section to do the simplification.

$$\left(\frac{m^{\frac{1}{2}}\,n^{-\frac{1}{3}}}{n^{\frac{2}{3}}\,m^{-\frac{7}{4}}}\right)^{-\frac{1}{6}} = \left(\frac{m^{\frac{1}{2}}\,m^{\frac{7}{4}}}{n^{\frac{2}{3}}\,n^{\frac{1}{3}}}\right)^{-\frac{1}{6}} = \left(\frac{m^{\frac{9}{4}}}{n^1}\right)^{-\frac{1}{6}} = \left(\frac{n}{m^{\frac{9}{4}}}\right)^{\frac{1}{6}} = \boxed{\frac{n^{\frac{1}{6}}}{m^{\frac{3}{8}}}}$$

1.3 Radicals

1. Write the following expression in exponential form.

$$\sqrt[7]{y}$$

Solution

All this problem is asking us to do is basically use the definition of the radical notation and write this in exponential form instead of radical form.

$$\boxed{y^{\frac{1}{7}}}$$

2. Write the following expression in exponential form.

$$\sqrt[3]{x^2}$$

Solution

All this problem is asking us to do is basically use the definition of the radical notation and write this in exponential form instead of radical form.

$$\boxed{\left(x^2\right)^{\frac{1}{3}}}$$

3. Write the following expression in exponential form.

$$\sqrt[6]{ab}$$

Solution

All this problem is asking us to do is basically use the definition of the radical notation and write this in exponential form instead of radical form.

$$\boxed{(ab)^{\frac{1}{6}}}$$

Be careful with parenthesis here! Recall that the only thing that gets the exponent is the term immediately to the left of the exponent. So, if we'd dropped parenthesis we'd get,

$$ab^{\frac{1}{6}} = a\left(b^{\frac{1}{6}}\right) = a\ \sqrt[6]{b}$$

which is most definitely not what we started with. The only way to make sure that we understand that both the a and the b were under the radical is to use parenthesis as we did above.

4. Write the following expression in exponential form.

$$\sqrt{w^2v^3}$$

Solution

All this problem is asking us to do is basically use the definition of the radical notation and write this in exponential form instead of radical form.

$$\boxed{\left(w^2v^3\right)^{\frac{1}{2}}}$$

Recall that when no index is written on the radical it is assumed to be 2.

Also, be careful with parenthesis here! Recall that the only thing that gets the exponent is the term immediately to the left of the exponent and so we need parenthesis on the whole thing to make sure that we understand that both terms were under the root.

5. Evaluate : $\sqrt[4]{81}$

Hint

Recall that the easiest way to evaluate radicals is to convert to exponential form and then also recall that we evaluated exponential forms in the Rational Exponent section.

Solution

All we need to do here is to convert this to exponential form and then recall that we learned how to evaluate the exponential form in the Rational Exponent section.

$$\sqrt[4]{81} = 81^{\frac{1}{4}} = \boxed{3} \qquad \text{because} \qquad 3^4 = 81$$

6. Evaluate : $\sqrt[3]{-512}$

Hint

Recall that the easiest way to evaluate radicals is to convert to exponential form and then also recall that we evaluated exponential forms in the Rational Exponent section.

Solution

All we need to do here is to convert this to exponential form and then recall that we learned how to evaluate the exponential form in the Rational Exponent section.

$$\sqrt[3]{-512} = (-512)^{\frac{1}{3}} = \boxed{-8} \qquad \text{because} \qquad (-8)^3 = -512$$

---

7. Evaluate : $\sqrt[3]{1000}$

Hint

Recall that the easiest way to evaluate radicals is to convert to exponential form and then also recall that we evaluated exponential forms in the Rational Exponent section.

Solution

All we need to do here is to convert this to exponential form and then recall that we learned how to evaluate the exponential form in the Rational Exponent section.

$$\sqrt[3]{1000} = 1000^{\frac{1}{3}} = \boxed{10} \qquad \text{because} \qquad 10^3 = 1000$$

---

8. Simplify the following expression. Assume that x is positive.

$$\sqrt[3]{x^8}$$

Step 1

Recall that by simplify we mean we want to put the expression in simplified radical form (which we defined in the notes for this section).

To do this for this expression we'll need to write the radicand as,

$$x^8 = x^6x^2 = \left(x^2\right)^3 x^2$$

Step 2

Now that we've gotten the radicand rewritten it's easy to deal with the radical and get the expression in simplified radical form.

$$\sqrt[3]{x^8} = \sqrt[3]{\left(x^2\right)^3 x^2} = \sqrt[3]{\left(x^2\right)^3}\sqrt[3]{x^2} = \boxed{x^2\ \sqrt[3]{x^2}}$$

9. Simplify the following expression. Assume that y is positive.

$$\sqrt{8y^3}$$

Step 1

Recall that by simplify we mean we want to put the expression in simplified radical form (which we defined in the notes for this section).

To do this for this expression we'll need to write the radicand as,

$$8y^3 = \left(4y^2\right)(2y)$$

Step 2

Now that we've gotten the radicand rewritten it's easy to deal with the radical and get the expression in simplified radical form.

$$\sqrt{8y^3} = \sqrt{\left(4y^2\right)(2y)} = \sqrt{4y^2}\sqrt{2y} = \boxed{2y\ \sqrt{2y}}$$

10. Simplify the following expression. Assume that x, y and z are positive.

$$\sqrt[4]{x^7 y^{20} z^{11}}$$

Step 1

Recall that by simplify we mean we want to put the expression in simplified radical form (which we defined in the notes for this section).

To do this for this expression we'll need to write the radicand as,

$$x^7 y^{20} z^{11} = x^4 y^{20} z^8 x^3 z^3 = x^4 \left(y^5\right)^4 \left(z^2\right)^4 x^3 z^3$$

Step 2

Now that we've gotten the radicand rewritten it's easy to deal with the radical and get the expression in simplified radical form.

$$\sqrt[4]{x^7 y^{20} z^{11}} = \sqrt[4]{x^4 (y^5)^4 (z^2)^4 x^3 z^3} = \sqrt[4]{x^4 (y^5)^4 (z^2)^4}\ \ \sqrt[4]{x^3 z^3} = \boxed{x\, y^5 z^2\ \ \sqrt[4]{x^3 z^3}}$$

11. Simplify the following expression. Assume that x, y and z are positive.

$$\sqrt[3]{54 x^6 y^7 z^2}$$

Step 1

Recall that by simplify we mean we want to put the expression in simplified radical form (which we defined in the notes for this section).

To do this for this expression we'll need to write the radicand as,

$$54 x^6 y^7 z^2 = \left(27 x^6 y^6\right)\left(2 y^1 z^2\right) = 3^3 \left(x^2\right)^3 \left(y^2\right)^3 \left(2 y z^2\right)$$

Step 2

Now that we've gotten the radicand rewritten it's easy to deal with the radical and get the expression in simplified radical form.

$$\sqrt[3]{54x^6y^7z^2} = \sqrt[3]{3^3\left(x^2\right)^3\left(y^2\right)^3\left(2yz^2\right)} = \sqrt[3]{3^3\left(x^2\right)^3\left(y^2\right)^3}\,\sqrt[3]{2yz^2} = \boxed{3x^2y^2\ \sqrt[3]{2yz^2}}$$

---

12. Simplify the following expression. Assume that x, y and z are positive.

$$\sqrt[4]{4x^3y}\ \sqrt[4]{8x^2y^3z^5}$$

Step 1

Remember that when we have a product of two radicals with the same index in an expression we first need to combine them into one root before we start the simplification process.

$$\sqrt[4]{4x^3y}\ \sqrt[4]{8x^2y^3z^5} = \sqrt[4]{\left(4x^3y\right)\left(8x^2y^3z^5\right)} = \sqrt[4]{32x^5y^4z^5}$$

Step 2

Now that the expression has been written as a single radical we can proceed as we did in the earlier problems.

The radicand can be written as,

$$32x^5y^4z^5 = \left(2^4x^4y^4z^4\right)(2xz)$$

Step 3

Now that we've gotten the radicand rewritten it's easy to deal with the radical and get the expression in simplified radical form.

$$\sqrt[4]{4x^3y}\ \sqrt[4]{8x^2y^3z^5} = \sqrt[4]{32x^5y^4z^5} = \sqrt[4]{2^4x^4y^4z^4}\ \sqrt[4]{2xz} = \boxed{2xyz\ \sqrt[4]{2xz}}$$

---

13. Multiply the following expression. Assume that x is positive.

$$\sqrt{x}\left(4-3\sqrt{x}\right)$$

Solution

All we need to do here is do the multiplication so here is that.

$$\sqrt{x}\left(4-3\sqrt{x}\right)=4\sqrt{x}-3\sqrt{x}\left(\sqrt{x}\right)=4\sqrt{x}-3\sqrt{x^2}=\boxed{4\sqrt{x}-3x}$$

Don't forget to simplify any resulting roots that can be. That is an often missed part of these problems.

---

14. Multiply the following expression. Assume that x is positive.

$$\left(2\sqrt{x}+1\right)\left(3-4\sqrt{x}\right)$$

Solution

All we need to do here is do the multiplication so here is that.

$$\left(2\sqrt{x}+1\right)\left(3-4\sqrt{x}\right)=6\sqrt{x}-8\sqrt{x}\left(\sqrt{x}\right)+3-4\sqrt{x}=3+2\sqrt{x}-8\sqrt{x^2}=\boxed{3+2\sqrt{x}-8x}$$

Don't forget to simplify any resulting roots that can be. That is an often missed part of these problems.

---

15. Multiply the following expression. Assume that x is positive.

$$\left(\sqrt[3]{x}+2\sqrt[3]{x^2}\right)\left(4-\sqrt[3]{x^2}\right)$$

Solution

All we need to do here is do the multiplication so here is that.

$$\begin{aligned}\left(\sqrt[3]{x}+2\sqrt[3]{x^2}\right)\left(4-\sqrt[3]{x^2}\right)&=4\sqrt[3]{x}-\sqrt[3]{x}\sqrt[3]{x^2}+8\sqrt[3]{x^2}-2\sqrt[3]{x^2}\sqrt[3]{x^2}\\&=4\sqrt[3]{x}-\sqrt[3]{x^3}+8\sqrt[3]{x^2}-2\sqrt[3]{x^4}\\&=4\sqrt[3]{x}-\sqrt[3]{x^3}+8\sqrt[3]{x^2}-2\sqrt[3]{x^3}\sqrt[3]{x}\\&=\boxed{4\sqrt[3]{x}-x+8\sqrt[3]{x^2}-2x\sqrt[3]{x}}\end{aligned}$$

Don't forget to simplify any resulting roots that can be. That is an often missed part of these problems and when dealing with roots other than square roots there can be quite a bit of work in the simplification process as we saw with this problem.

16. Rationalize the denominator. Assume that x is positive.

$$\frac{6}{\sqrt{x}}$$

Solution

For this problem we need to multiply the numerator and denominator by $\sqrt{x}$ in order to rationalize the denominator.

$$\frac{6}{\sqrt{x}} = \frac{6}{\sqrt{x}}\frac{\sqrt{x}}{\sqrt{x}} = \frac{6\sqrt{x}}{\sqrt{x^2}} = \boxed{\frac{6\sqrt{x}}{x}}$$

17. Rationalize the denominator. Assume that x is positive.

$$\frac{9}{\sqrt[3]{2x}}$$

Solution

For this problem we need to multiply the numerator and denominator by $\sqrt[3]{(2x)^2}$ in order to rationalize the denominator.

$$\frac{9}{\sqrt[3]{2x}} = \frac{9}{\sqrt[3]{2x}}\frac{\sqrt[3]{(2x)^2}}{\sqrt[3]{(2x)^2}} = \frac{9\sqrt[3]{(2x)^2}}{\sqrt[3]{(2x)^3}} = \frac{9\sqrt[3]{(2x)^2}}{2x} = \boxed{\frac{9\sqrt[3]{4x^2}}{2x}}$$

18. Rationalize the denominator. Assume that x and y are positive.

$$\frac{4}{\sqrt{x}+2\sqrt{y}}$$

Solution

For this problem we need to multiply the numerator and denominator by $\sqrt{x}-2\sqrt{y}$ in order to rationalize the denominator.

$$\frac{4}{\sqrt{x}+2\sqrt{y}} = \frac{4}{\sqrt{x}+2\sqrt{y}}\frac{\sqrt{x}-2\sqrt{y}}{\sqrt{x}-2\sqrt{y}} = \frac{4\left(\sqrt{x}-2\sqrt{y}\right)}{\left(\sqrt{x}+2\sqrt{y}\right)\left(\sqrt{x}-2\sqrt{y}\right)} = \boxed{\frac{4\sqrt{x}-8\sqrt{y}}{x-4y}}$$

19. Rationalize the denominator. Assume that x is positive.

$$\frac{10}{3-5\sqrt{x}}$$

Solution

For this problem we need to multiply the numerator and denominator by $3+5\sqrt{x}$ in order to rationalize the denominator.

$$\frac{10}{3-5\sqrt{x}} = \frac{10}{3-5\sqrt{x}}\frac{3+5\sqrt{x}}{3+5\sqrt{x}} = \frac{10\left(3+5\sqrt{x}\right)}{\left(3-5\sqrt{x}\right)\left(3+5\sqrt{x}\right)} = \boxed{\frac{30+50\sqrt{x}}{9-25x}}$$

1.4 Polynomials

1. Perform the indicated operation and identify the degree of the result.

Add $4x^3 - 2x^2 + 1$ to $7x^2 + 12x$

Step 1

Here is the operation we're being asked to perform.

$$\left(4x^3 - 2x^2 + 1\right) + \left(7x^2 + 12x\right)$$

Note that the parenthesis are only there to illustrate how each polynomial is being used in the indicated operation and are not needed (or used) in general.

Here's the result of the operation.

$$\left(4x^3 - 2x^2 + 1\right) + \left(7x^2 + 12x\right) = \boxed{4x^3 + 5x^2 + 12x + 1}$$

Step 2

Remember the degree of a polynomial is just the largest exponent in the polynomial and so the degree of the result of this operation is **three**.

2. Perform the indicated operation and identify the degree of the result.

Subtract $4z^6 - 3z^2 + 2z$ from $-10z^6 + 7z^2 - 8$

Step 1

Here is the operation we're being asked to perform.

$$-10z^6 + 7z^2 - 8 - \left(4z^6 - 3z^2 + 2z\right)$$

Be careful with the order here! We are subtracting the first polynomial from the second and that implies the order we've got here. Also be careful with the parenthesis on the second polynomial. We are subtracting the whole polynomial and so we need to have the parenthesis to do that.

Here's the result of the operation.

$$-10z^6 + 7z^2 - 8 - \left(4z^6 - 3z^2 + 2z\right) = -10z^6 + 7z^2 - 8 - 4z^6 + 3z^2 - 2z$$
$$= \boxed{-14z^6 + 10z^2 - 2z - 8}$$

Step 2

emember the degree of a polynomial is just the largest exponent in the polynomial and so the degree of the result of this operation is **six**.

3. Perform the indicated operation and identify the degree of the result.

Subtract $-3x^2 + 7x + 8$ from $x^4 + 7x^3 - 12x - 1$

Step 1

Here is the operation we're being asked to perform.

$$x^4 + 7x^3 - 12x - 1 - \left(-3x^2 + 7x + 8\right)$$

Be careful with the order here! We are subtracting the first polynomial from the second and that implies the order we've got here. Also be careful with the parenthesis on the second polynomial. We are subtracting the whole polynomial and so we need to have the parenthesis to do that.

Here's the result of the operation.

$$x^4 + 7x^3 - 12x - 1 - \left(-3x^2 + 7x + 8\right) = x^4 + 7x^3 - 12x - 1 + 3x^2 - 7x - 8$$
$$= \boxed{x^4 + 7x^3 + 3x^2 - 19x - 9}$$

Step 2

Remember the degree of a polynomial is just the largest exponent in the polynomial and so the degree of the result of this operation is **four**.

4. Perform the indicated operation and identify the degree of the result.

$$12y\left(3y^4 - 7y^2 + 1\right)$$

Step 1

All we need to do is multiply the 12 y through the second polynomial. Here is the result of doing that.

$$12y\left(3y^4 - 7y^2 + 1\right) = \boxed{36y^5 - 84y^3 + 12y}$$

Step 2

Remember the degree of a polynomial is just the largest exponent in the polynomial and so the degree of the result of this operation is **five**.

---

5. Perform the indicated operation and identify the degree of the result.

$$(3x+1)\left(2 - 9x^2\right)$$

Step 1

All we need to do is foil out the two polynomials. Here is the result of doing that.

$$(3x+1)\left(2 - 9x^2\right) = \boxed{2 + 6x - 9x^2 - 27x^3}$$

Step 2

Remember the degree of a polynomial is just the largest exponent in the polynomial and so the degree of the result of this operation is **three**.

---

6. Perform the indicated operation and identify the degree of the result.

$$\left(w^2+2\right)\left(3w^2+w\right)$$

Step 1

All we need to do is foil out the two polynomials. Here is the result of doing that.

$$\left(w^2+2\right)\left(3w^2+w\right) = \boxed{3w^4+w^3+6w^2+2w}$$

Step 2

Remember the degree of a polynomial is just the largest exponent in the polynomial and so the degree of the result of this operation is **four**.

7. Perform the indicated operation and identify the degree of the result.

$$\left(4x^6-3x\right)\left(4x^6+3x\right)$$

Step 1

All we need to do is foil out the two polynomials. Here is the result of doing that.

$$\left(4x^6-3x\right)\left(4x^6+3x\right) = \boxed{16x^{12}-9x^2}$$

Step 2

Remember the degree of a polynomial is just the largest exponent in the polynomial and so the degree of the result of this operation is **twelve**.

8. Perform the indicated operation and identify the degree of the result.

$$3\left(10-4y^3\right)^2$$

Step 1

Remember that this is just another way of writing,

$$3\left(10-4y^3\right)^2 = 3\left(10-4y^3\right)\left(10-4y^3\right)$$

Now all we need to do is foil out the two polynomials. Here is the result of doing that.

$$3\left(10-4y^3\right)^2 = 3\left(10-4y^3\right)\left(10-4y^3\right) = 3\left(100-80y^3+16y^6\right) = \boxed{300-240y^3+48y^6}$$

Be careful with dealing with the three! Make sure you take care of the exponent first (*i.e.* make sure you multiply out the product first) before you multiply the three through the result.

Step 2

Remember the degree of a polynomial is just the largest exponent in the polynomial and so the degree of the result of this operation is **six**.

9. Perform the indicated operation and identify the degree of the result.

$$\left(x^2+x-2\right)\left(3x^2-8x-7\right)$$

Step 1

Remember that the foil method only works for binomials and these are both trinomials (*i.e.* they each have three terms).

So, all we need to do is multiply each term in the second polynomial by each term in the first polynomial. Here is the result of doing that.

$$\begin{aligned}\left(x^2+x-2\right)\left(3x^2-8x-7\right) &= 3x^4-8x^3-7x^2+3x^3-8x^2-7x-6x^2+16x+14\\ &= \boxed{3x^4-5x^3-21x^2+9x+14}\end{aligned}$$

Step 2

Remember the degree of a polynomial is just the largest exponent in the polynomial and so the degree of the result of this operation is **four**.

10. Perform the indicated operation and identify the degree of the result.

Subtract $3\left(x^2+1\right)^2$ from $6x^3-9x^2-13x-4$

Step 1

Here is the operation we're being asked to perform.

$$6x^3-9x^2-13x-4-3\left(x^2+1\right)^2$$

Now, before we actually do the subtraction we need to actually multiply out the second term before we do the subtraction. Here are the results of all these operations.

$$\begin{aligned}6x^3-9x^2-13x-4-3\left(x^2+1\right)^2 &= 6x^3-9x^2-13x-4-3\left(x^4+2x^2+1\right)\\ &= 6x^3-9x^2-13x-4-3x^4-6x^2-3\\ &= \boxed{-3x^4+6x^3-15x^2-13x-7}\end{aligned}$$

Step 2

Remember the degree of a polynomial is just the largest exponent in the polynomial and so the degree of the result of this operation is **four**.

1.5 Factoring Polynomials

1. Factor out the greatest common factor from the following polynomial.

$$6x^7 + 3x^4 - 9x^3$$

Step 1

The first step is to identify the greatest common factor. In this case it looks like we can factor a 3 and an x^3 out of each term and so the greatest common factor is $3x^3$.

Step 2

Okay, now let's do the factoring.

$$6x^7 + 3x^4 - 9x^3 = \boxed{3x^3\left(2x^4 + x - 3\right)}$$

Don't forget to also identify any numbers in the greatest common factor as well. That can often greatly simplify the problem for later work (when we have later work for the problem anyway....).

2. Factor out the greatest common factor from the following polynomial.

$$a^3b^8 - 7a^{10}b^4 + 2a^5b^2$$

Step 1

The first step is to identify the greatest common factor. In this case it looks like we can factor an a^3 and a b^2 out of each term and so the greatest common factor is a^3b^2 .

Step 2

Okay, now let's do the factoring.

$$a^3b^8 - 7a^{10}b^4 + 2a^5b^2 = \boxed{a^3b^2\left(b^6 - 7a^7b^2 + 2a^2\right)}$$

3. Factor out the greatest common factor from the following polynomial.

$$2x\left(x^2+1\right)^3 - 16\left(x^2+1\right)^5$$

Step 1

The first step is to identify the greatest common factor. In this case it looks like we can factor a 2 and an $\left(x^2+1\right)^3$ out of each term and so the greatest common factor is $2\left(x^2+1\right)^3$.

Step 2

Okay, now let's do the factoring.

$$2x\left(x^2+1\right)^3-16\left(x^2+1\right)^5=\boxed{2\left(x^2+1\right)^3\left(x-8\left(x^2+1\right)^2\right)}$$

Don't get excited if the greatest common factor has more "complicated" terms in it as this one did. The greatest common factor won't always be just variables to powers.

4. Factor out the greatest common factor from the following polynomial.

$$x^2\left(2-6x\right)+4x\left(4-12x\right)$$

Step 1

The first step is to identify the greatest common factor and in this case we'll need to be a little careful. If we just do a quick glance we might be tempted to just say the greatest common factor is just x since there is clearly an x in both terms.

However, notice that we can factor a 2 out of the $4-12x$ in the second term to get,

$$x^2\left(2-6x\right)+4x\left(4-12x\right)=x^2\left(2-6x\right)+8x\left(2-6x\right)$$

Upon doing this we see that not only do we have an x in both terms we also have a $2-6x$ in both terms and so the greatest common factor in this case is $x\left(2-6x\right)$.

Step 2

Okay, now let's do the factoring.

$$x^2\left(2-6x\right)+4x\left(4-12x\right)=x^2\left(2-6x\right)+8x\left(2-6x\right)=\boxed{x\left(2-6x\right)\left(x+8\right)}$$

Sometimes we need to do a little "pre factoring" work on a polynomial in order to determine just what the greatest common factor is. It won't happen often, but it does need to be done often enough that we can't forget about it.

5. Factor the following polynomial by grouping.

$$7x + 7x^3 + x^4 + x^6$$

Step 1

The first step here is to group the first two term and the last two terms as follows.

$$\left(7x + 7x^3\right) + \left(x^4 + x^6\right)$$

Step 2

We can now see that we can factor a 7 x out of the first grouping and an x^4 out of the second grouping. Doing this gives,

$$7x + 7x^3 + x^4 + x^6 = 7x\left(1 + x^2\right) + x^4\left(1 + x^2\right)$$

Step 3

Finally, we see that we can factor an $x\left(1 + x^2\right)$ out of both of the new terms to get,

$$7x + 7x^3 + x^4 + x^6 = \boxed{x\left(1 + x^2\right)\left(7 + x^3\right)}$$

6. Factor the following polynomial by grouping.

$$18x + 33 - 6x^4 - 11x^3$$

Step 1

The first step here is to group the first two term and the last two terms as follows.

$$(18x + 33) - (6x^4 + 11x^3)$$

Be careful with the last grouping. Because both of the terms were negative we needed to factor out an "-" as we did the grouping.

Step 2

We can now see that we can factor a 3 out of the first grouping and an x^3 out of the second grouping. Doing this gives,

$$18x + 33 - 6x^4 - 11x^3 = 3(6x + 11) - x^3(6x + 11)$$

Step 3

Finally, we see that we can factor a $6x + 11$ out of both of the new terms to get,

$$18x + 33 - 6x^4 - 11x^3 = \boxed{(6x + 11)(3 - x^3)}$$

---

7. Factor the following polynomial.

$$x^2 - 2x - 8$$

Step 1

The initial form for the factoring will be,

$$(x + \_\_)(x + \_\_)$$

and the factors of -8 are,

$$(-1)(8) \qquad (1)(-8) \qquad (-2)(4) \qquad (2)(-4)$$

Step 2

Now, recalling that we need the pair of factors from the above list that will add to get -2. So, we can see that the correct factoring will then be,

$$x^2 - 2x - 8 = \boxed{(x-4)(x+2)}$$

---

8. Factor the following polynomial.

$$z^2 - 10z + 21$$

Step 1

The initial form for the factoring will be,

$$(z + \_\_)(z + \_\_)$$

and the factors of 21 are,

$$(-1)(-21) \qquad (1)(21) \qquad (-3)(-7) \qquad (3)(7)$$

Step 2

Now, recalling that we need the pair of factors from the above list that will add to get -10. So, we can see that the correct factoring will then be,

$$z^2 - 10z + 21 = \boxed{(z-3)(z-7)}$$

---

9. Factor the following polynomial.

$$y^2 + 16y + 60$$

Step 1

The initial form for the factoring will be,

$$(y + \_\_)(y + \_\_)$$

and the factors of 60 are,

$$\begin{array}{cccccc} (-1)(-60) & (-2)(-30) & (-3)(-20) & (-4)(-15) & (-5)(-12) & (-6)(-10) \\ (1)(60) & (2)(30) & (3)(20) & (4)(15) & (5)(12) & (6)(10) \end{array}$$

Sometimes there are a lot of factors that we need to deal with. As you get more practice you will start to be able to do most of this in your head and won't need to actually write all of the factors down.

Step 2

Now, recalling that we need the pair of factors from the above list that will add to get 16. So, we can see that the correct factoring will then be,

$$y^2 + 16y + 60 = \boxed{(y+6)(y+10)}$$

---

10. Factor the following polynomial.

$$5x^2 + 14x - 3$$

Step 1

There are only two positive factors of 5 so the initial form for the factoring will be,

$$(5x + \_)(x + \_)$$

and the factors of -3 are,

$$(-1)(3) \qquad (1)(-3)$$

Step 2

After some trial and error we see that the correct factoring will then be,

$$5x^2 + 14x - 3 = \boxed{(5x-1)(x+3)}$$

---

11. Factor the following polynomial.

$$6t^2 - 19t - 7$$

Step 1

There are two sets of positive factors of 6 and so we will have one of the two following possible initial forms for the factoring.

$$(2t + \_)(3t + \_) \qquad\qquad (6t + \_)(t + \_)$$

and the factors of -7 are,

$$(-1)(7) \qquad (1)(-7)$$

Step 2

After some trial and error we see that the correct factoring will then be,

$$6t^2 - 19t - 7 = \boxed{(2t-7)(3t+1)}$$

12. Factor the following polynomial.

$$4z^2 + 19z + 12$$

Step 1

There are two sets of positive factors of 4 and so we will have one of the two following possible initial forms for the factoring.

$$(2z + \_)(2z + \_) \qquad (4z + \_)(z + \_)$$

and the factors of 12 are,

$$(-1)(-12) \quad (-2)(-6) \quad (-3)(-4) \quad (1)(12) \quad (2)(6) \quad (3)(4)$$

Step 2

After some trial and error we see that the correct factoring will then be,

$$4z^2 + 19z + 12 = \boxed{(4z+3)(z+4)}$$

13. Factor the following polynomial.

$$x^2 + 14x + 49$$

Solution

We can do this in the manner of the previous problems if we wanted to. On the other hand we can notice that the constant is a perfect square and that $2(7) = 14$ and so we can see that this is one of the special forms.

Therefore, the factoring of this polynomial is,

$$x^2 + 14x + 49 = \boxed{(x + 7)^2}$$

Note that while you don't need necessarily need to know the special forms if you do and can easily recognize them it will make the factoring easier.

14. Factor the following polynomial.

$$4w^2 - 25$$

Solution

We can do this in the manner of the previous problems if we wanted to. On the other hand we can notice that we have a difference of perfect squares and so this is one of the special forms.

Therefore, the factoring of this polynomial is,

$$4w^2 - 25 = \boxed{(2w - 5)(2w + 5)}$$

Note that while you don't need necessarily need to know the special forms if you do and can easily recognize them it will make the factoring easier.

15. Factor the following polynomial.

$$81x^2 - 36x + 4$$

Solution

We can do this in the manner of the previous problems if we wanted to. On the other hand we can notice that the constant is a perfect square and the coefficient of the x^2 is also a perfect square. We can also notice that that $2(9)(2) = 36$ and so we can see that this is one of the special forms.

Therefore, the factoring of this polynomial is,

$$81x^2 - 36x + 4 = \boxed{(9x - 2)^2}$$

Note that while you don't need necessarily need to know the special forms if you do and can easily recognize them it will make the factoring easier.

16. Factor the following polynomial.

$$x^6 + 3x^3 - 4$$

Step 1

Don't let the fact that this polynomial is not quadratic worry you. Just because it's not a quadratic polynomial doesn't mean that we can't factor it.

For this polynomial we can see that $\left(x^3\right)^2 = x^6$ and so it looks like we can factor this into the form,

$$\left(x^3 + \_\right)\left(x^3 + \_\right)$$

At this point all we need to do is proceed as we did with the quadratics we were factoring above.

Step 2

After writing down the factors of -4 we can see that we need to have the following factoring.

$$x^6 + 3x^3 - 4 = \left(x^3 + 4\right)\left(x^3 - 1\right)$$

Step 3

Now, we need to be careful here. Sometimes these will have further factoring we can do. In this case we can see that the second factor is a difference of perfect cubes and we have a formula for factoring a difference of perfect cubes.

Therefore, the factoring of this polynomial is,

$$x^6 + 3x^3 - 4 = \left(x^3 + 4\right)\left(x^3 - 1\right) = \boxed{\left(x^3 + 4\right)\left(x - 1\right)\left(x^2 + x + 1\right)}$$

---

17. Factor the following polynomial.

$$3z^5 - 17z^4 - 28z^3$$

Step 1

Don't let the fact that this polynomial is not quadratic worry you. Just because it's not a quadratic polynomial doesn't mean that we can't factor it.

For this polynomial note that we can factor a z^3 out of each term to get,

$$3z^5 - 17z^4 - 28z^3 = z^3\left(3z^2 - 17z - 28\right)$$

Step 2

Now, notice that the second factor is a quadratic and we know how to factor these. So, it looks like the form of the factoring should be,

$$3z^5 - 17z^4 - 28z^3 = z^3\left(3z + \_\right)\left(z + \_\right)$$

Step 3

Finally, once we write down the factors of the -28 we can see that the factoring of this polynomial is,

$$3z^5 - 17z^4 - 28z^3 = \boxed{z^3\left(3z + 4\right)\left(z - 7\right)}$$

18. Factor the following polynomial.

$$2x^{14} - 512x^6$$

Step 1

Don't let the fact that this polynomial is not quadratic worry you. Just because it's not a quadratic polynomial doesn't mean that we can't factor it.

For this polynomial note that we can factor a $2x^6$ out of each term to get,

$$2x^{14} - 512x^6 = 2x^6\left(x^8 - 256\right)$$

Step 2

Now, notice that the second factor is a difference of perfect squares and so we can further factor this as,

$$2x^{14} - 512x^6 = 2x^6\left(x^4 + 16\right)\left(x^4 - 16\right)$$

Step 3

Next, we can see that the third term is once again a difference of perfect squares and so can also be factored. After doing that the factoring of this polynomial is,

$$2x^{14} - 512x^6 = 2x^6\left(x^4 + 16\right)\left(x^2 + 4\right)\left(x^2 - 4\right)$$

Step 4

Finally, we can see that we can do one more factoring on the last factor.

$$2x^{14} - 512x^6 = \boxed{2x^6\left(x^4 + 16\right)\left(x^2 + 4\right)\left(x + 2\right)\left(x - 2\right)}$$

Do not get too excited about polynomials that have lots of factoring in them. They will happen on occasion so don't worry about it when they do.

1.6 Rational Expressions

1. Reduce the following rational expression to lowest terms.

$$\frac{x^2-6x-7}{x^2-10x+21}$$

Step 1

First, we need to factor the numerator and denominator as much as we can. Doing that gives,

$$\frac{x^2-6x-7}{x^2-10x+21}=\frac{(x-7)(x+1)}{(x-7)(x-3)}$$

Step 2

Now all we need to do is cancel all the factors that we can in order to reduce the rational expression to lowest terms.

$$\frac{x^2-6x-7}{x^2-10x+21}=\boxed{\frac{x+1}{x-3}}$$

2. Reduce the following rational expression to lowest terms.

$$\frac{x^2+6x+9}{x^2-9}$$

Step 1

First, we need to factor the numerator and denominator as much as we can. Doing that gives,

$$\frac{x^2+6x+9}{x^2-9}=\frac{(x+3)^2}{(x+3)(x-3)}$$

Step 2

Now all we need to do is cancel all the factors that we can in order to reduce the rational expression to lowest terms.

$$\frac{x^2+6x+9}{x^2-9} = \boxed{\frac{x+3}{x-3}}$$

3. Reduce the following rational expression to lowest terms.

$$\frac{2x^2-x-28}{20-x-x^2}$$

Step 1

First, we need to factor the numerator and denominator as much as we can. Doing that gives,

$$\frac{2x^2-x-28}{20-x-x^2} = \frac{2x^2-x-28}{-\left(x^2+x-20\right)} = \frac{(2x+7)(x-4)}{-(x+5)(x-4)}$$

Notice that in order to make factoring the denominator somewhat easier we first factored a minus sign out of the denominator.

Step 2

Now all we need to do is cancel all the factors that we can in order to reduce the rational expression to lowest terms.

$$\frac{2x^2-x-28}{20-x-x^2} = \boxed{-\frac{2x+7}{x+5}}$$

Recall that the minus sign in the denominator can be put out in front of the rational expression if we choose to put it there (as we did here).

4. Perform the indicated operation in the following expression and reduce the answer to lowest terms.

$$\frac{x^2+5x-24}{x^2+6x+8}\cdot\frac{x^2+4x+4}{x^2-3x}$$

Step 1

So, we first need to factor each of the polynomials as much as possible.

$$\frac{(x+8)(x-3)}{(x+4)(x+2)}\cdot\frac{(x+2)^2}{x(x-3)}=\frac{(x+8)}{(x+4)}\cdot\frac{(x+2)}{x}$$

Step 2

Finally, just multiply the two terms together. Doing this gives,

$$\frac{x^2+5x-24}{x^2+6x+8}\cdot\frac{x^2+4x+4}{x^2-3x}=\boxed{\frac{(x+8)(x+2)}{x(x+4)}}$$

5. Perform the indicated operation in the following expression and reduce the answer to lowest terms.

$$\frac{x^2-49}{2x^2-3x-5}\div\frac{x^2-x-42}{x^2+7x+6}$$

Step 1

So, we first need to do is convert this into a product.

$$\frac{x^2-49}{2x^2-3x-5}\div\frac{x^2-x-42}{x^2+7x+6}=\frac{x^2-49}{2x^2-3x-5}\cdot\frac{x^2+7x+6}{x^2-x-42}$$

Make sure that you don't do the factoring and canceling until you've converted the division to a product.

Step 2

Now we can factor each of the terms as much as possible to get,

$$\frac{x^2-49}{2x^2-3x-5}\div\frac{x^2-x-42}{x^2+7x+6}=\frac{(x-7)(x+7)}{(2x-5)(x+1)}\cdot\frac{(x+1)(x+6)}{(x-7)(x+6)}$$

Step 3

Finally cancel as much as possible to reduce to lowest terms and do the product.

$$\frac{x^2-49}{2x^2-3x-5} \div \frac{x^2-x-42}{x^2+7x+6} = \boxed{\frac{x+7}{2x-5}}$$

6. Perform the indicated operation in the following expression and reduce the answer to lowest terms.

$$\frac{x^2-2x-8}{2x^2-8x-24} \div \frac{x^2-9x+20}{x^2-11x+30}$$

Step 1

So, we first need to do is convert this into a product.

$$\frac{x^2-2x-8}{2x^2-8x-24} \div \frac{x^2-9x+20}{x^2-11x+30} = \frac{x^2-2x-8}{2x^2-8x-24} \cdot \frac{x^2-11x+30}{x^2-9x+20}$$

Make sure that you don't do the factoring and canceling until you've converted the division to a product.

Step 2

Now we can factor each of the terms as much as possible to get,

$$\frac{x^2-2x-8}{2x^2-8x-24} \div \frac{x^2-9x+20}{x^2-11x+30} = \frac{(x-4)(x+2)}{2(x+2)(x-6)} \cdot \frac{(x-5)(x-6)}{(x-5)(x-4)}$$

Step 3

Finally cancel as much as possible to reduce to lowest terms and do the product.

$$\frac{x^2-2x-8}{2x^2-8x-24} \div \frac{x^2-9x+20}{x^2-11x+30} = \boxed{\frac{1}{2}}$$

Don't worry if all the variables end up cancelling out after you are done reducing to lowest terms. It will happen on occasion so don't worry about it when it does.

7. Perform the indicated operation in the following expression and reduce the answer to lowest

terms.

$$\frac{\frac{3}{x+1}}{\frac{x+4}{x^2+11x+10}}$$

Step 1

This is just a division and so let's first convert it to a product.

$$\frac{\frac{3}{x+1}}{\frac{x+4}{x^2+11x+10}} = \frac{3}{x+1}\cdot\frac{x^2+11x+10}{x+4}$$

Step 2

Now we can factor each of the second term as much as possible to get,

$$\frac{\frac{3}{x+1}}{\frac{x+4}{x^2+11x+10}} = \frac{3}{x+1}\cdot\frac{(x+1)(x+10)}{x+4}$$

Step 3

Now cancel as much as possible to reduce to lowest terms and do the product.

$$\frac{\frac{3}{x+1}}{\frac{x+4}{x^2+11x+10}} = \boxed{\frac{3(x+10)}{x+4}}$$

8. Perform the indicated operation in the following expression.

$$\frac{3}{x-4}+\frac{x}{2x+7}$$

Step 1

We first need the least common denominator for this rational expression.

$$\text{lcd}: \ (x-4)(2x+7)$$

Step 2

Now multiply each term by an appropriate quantity to get the least common denominator into the denominator of each term.

$$\frac{3}{x-4}+\frac{x}{2x+7}=\frac{3(2x+7)}{(x-4)(2x+7)}+\frac{x(x-4)}{(2x+7)(x-4)}$$

Step 3

All we need to do now is do the addition and simplify the numerator of the result.

$$\frac{3}{x-4}+\frac{x}{2x+7}=\frac{3(2x+7)+x(x-4)}{(x-4)(2x+7)}=\frac{6x+21+x^2-4x}{(x-4)(2x+7)}=\boxed{\frac{x^2+2x+21}{(x-4)(2x+7)}}$$

9. Perform the indicated operation in the following expression.

$$\frac{2}{3x^2}-\frac{1}{9x^4}+\frac{2}{x+4}$$

Step 1

We first need the least common denominator for this rational expression.

$$\text{lcd}: 9x^4(x+4)$$

Step 2

Now multiply each term by an appropriate quantity to get the least common denominator into the denominator of each term.

$$\frac{2}{3x^2}-\frac{1}{9x^4}+\frac{2}{x+4}=\frac{2\left(3x^2\right)(x+4)}{3x^2\left(3x^2\right)(x+4)}-\frac{1\,(x+4)}{9x^4\,(x+4)}+\frac{2\left(9x^4\right)}{(x+4)\left(9x^4\right)}$$

Step 3

All we need to do now is do the subtraction and addition then simplify the numerator of the result.

$$\frac{2}{3x^2}-\frac{1}{9x^4}+\frac{2}{x+4}=\frac{6x^3+24x^2-(x+4)+18x^4}{9x^4\,(x+4)}=\boxed{\frac{18x^4+6x^3+24x^2-x-4}{9x^4\,(x+4)}}$$

10. Perform the indicated operation in the following expression.

$$\frac{x}{x^2+12x+36}-\frac{x-8}{x+6}$$

Step 1

We first need the least common denominator for this rational expression. However, before we get that we'll need to factor the denominator of the first term. Doing this gives,

$$\frac{x}{x^2+12x+36}-\frac{x-8}{x+6}=\frac{x}{(x+6)^2}-\frac{x-8}{x+6}$$

Step 2

The least common denominator is then,

$$\text{lcd}:\ (x+6)^2$$

Remember that we only take the highest power on each term in the denominator when setting up the least common denominator.

Step 3

Next, multiply each term by an appropriate quantity to get the least common denominator into the denominator of each term.

$$\frac{x}{x^2+12x+36}-\frac{x-8}{x+6}=\frac{x}{(x+6)^2}-\frac{(x-8)(x+6)}{(x+6)(x+6)}$$

Step 4

Finally, all we need to do is the subtraction and simplify the numerator of the result.

$$\frac{x}{x^2+12x+36}-\frac{x-8}{x+6}=\frac{x-(x-8)(x+6)}{(x+6)^2}=\frac{x-\left(x^2-2x-48\right)}{(x+6)^2}=\boxed{\frac{48+3x-x^2}{(x+6)^2}}$$

---

11. Perform the indicated operation in the following expression.

$$\frac{1}{x^2-13x+42}+\frac{x+1}{x-6}-\frac{x^2}{x-7}$$

Step 1

We first need the least common denominator for this rational expression. However, before we get that we'll need to factor the denominator of the first term. Doing this gives,

$$\frac{1}{x^2-13x+42}+\frac{x+1}{x-6}-\frac{x^2}{x-7}=\frac{1}{(x-6)(x-7)}+\frac{x+1}{x-6}-\frac{x^2}{x-7}$$

Step 2

The least common denominator is then,

$$\text{lcd}: \ (x-6)(x-7)$$

Remember that we only take the highest power on each term in the denominator when setting up the least common denominator.

Step 3

Next, multiply each term by an appropriate quantity to get the least common denominator into the denominator of each term.

$$\frac{1}{x^2-13x+42}+\frac{x+1}{x-6}-\frac{x^2}{x-7}=\frac{1}{(x-6)(x-7)}+\frac{(x+1)(x-7)}{(x-6)(x-7)}-\frac{x^2(x-6)}{(x-7)(x-6)}$$

Step 4

Finally, all we need to do is the addition and subtraction then simplify the numerator of the result.

$$\begin{aligned}\frac{1}{x^2-13x+42}+\frac{x+1}{x-6}-\frac{x^2}{x-7}&=\frac{1+(x+1)(x-7)-x^2(x-6)}{(x-6)(x-7)}\\&=\frac{1+x^2-6x-7-x^3+6x^2}{(x-6)(x-7)}=\boxed{\frac{-x^3+7x^2-6x-6}{(x-6)(x-7)}}\end{aligned}$$

12. Perform the indicated operation in the following expression.

$$\frac{x+10}{(3x+8)^3}+\frac{x}{(3x+8)^2}$$

Step 1

We first need the least common denominator for this rational expression.

lcd : $(3x+8)^3$

Remember that we only take the highest power on each term in the denominator when setting up the least common denominator.

Step 2

Now multiply each term by an appropriate quantity to get the least common denominator into the denominator of each term.

$$\frac{x+10}{(3x+8)^3}+\frac{x}{(3x+8)^2}=\frac{x+10}{(3x+8)^3}+\frac{x(3x+8)}{(3x+8)^2(3x+8)}$$

Step 3

All we need to do now is do the addition and simplify the numerator of the result.

$$\frac{x+10}{(3x+8)^3}+\frac{x}{(3x+8)^2}=\frac{x+10+3x^2+8x}{(3x+8)^3}=\boxed{\frac{3x^2+9x+10}{(3x+8)^3}}$$

1.7 Complex Numbers

1. Perform the indicated operation and write your answer in standard form.

$$(4 - 5i)(12 + 11i)$$

Hint

You know how to do the operation with polynomials so you can do the operation here! Just recall that you need to be careful to deal with any i ² that might happen to show up in the process.

Solution

We know how to multiply two polynomials and so we also know how to multiply two complex numbers. All we need to do is "foil" the two complex numbers to get,

$$(4 - 5i)(12 + 11i) = 48 + 44i - 60i - 55i^2 = 48 - 16i - 55i^2$$

All we need to do to finish the problem is to recall that $i^2 = -1$. Upon using this fact we can finish the problem.

$$(4 - 5i)(12 + 11i) = 48 - 16i - 55(-1) = \boxed{103 - 16i}$$

---

2. Perform the indicated operation and write your answer in standard form.

$$(-3 - i) - (6 - 7i)$$

Hint

You know how to do the operation with polynomials so you can do the operation here!

Solution

We know how to subtract two polynomials and so we also know how to subtract two complex numbers.

$$(-3 - i) - (6 - 7i) = -3 - i - 6 + 7i = \boxed{-9 + 6i}$$

---

3. Perform the indicated operation and write your answer in standard form.

$$(1 + 4i) - (-16 + 9i)$$

Hint

You know how to do the operation with polynomials so you can do the operation here!

Solution

We know how to subtract two polynomials and so we also know how to subtract two complex numbers.

$$(1 + 4i) - (-16 + 9i) = 1 + 4i + 16 - 9i = \boxed{17 - 5i}$$

---

4. Perform the indicated operation and write your answer in standard form.

$$8i\,(10 + 2i)$$

Hint

You know how to do the operation with polynomials so you can do the operation here! Just recall that you need to be careful to deal with any i^2 that might happen to show up in the process.

Solution

We know how to multiply two polynomials and so we also know how to multiply two complex numbers. All we need to do is distribute the 8 i to get,

$$8i\,(10 + 2i) = 80i + 16i^2$$

All we need to do to finish the problem is to recall that $i^2 = -1$. Upon using this fact we can finish the problem.

$$8i\,(10 + 2i) = 80i + 16\,(-1) = \boxed{-16 + 80i}$$

---

5. Perform the indicated operation and write your answer in standard form.

$$(-3-9i)(1+10i)$$

Hint

You know how to do the operation with polynomials so you can do the operation here! Just recall that you need to be careful to deal with any i^2 that might happen to show up in the process.

Solution

We know how to multiply two polynomials and so we also know how to multiply two complex numbers. All we need to do is "foil" the two complex numbers to get,

$$(-3-9i)(1+10i) = -3-30i-9i-90i^2$$

All we need to do to finish the problem is to recall that $i^2=-1$. Upon using this fact we can finish the problem.

$$(-3-9i)(1+10i) = -3-30i-9i-90(-1) = \boxed{87-39i}$$

---

6. Perform the indicated operation and write your answer in standard form.

$$(2+7i)(8+3i)$$

Hint

You know how to do the operation with polynomials so you can do the operation here! Just recall that you need to be careful to deal with any i^2 that might happen to show up in the process.

Solution

We know how to multiply two polynomials and so we also know how to multiply two complex numbers. All we need to do is "foil" the two complex numbers to get,

$$(2+7i)(8+3i) = 16+6i+56i+21i^2$$

All we need to do to finish the problem is to recall that $i^2=-1$. Upon using this fact we can finish the problem.

$$(2+7i)(8+3i) = 16+6i+56i+21(-1) = \boxed{-5+62i}$$

7. Perform the indicated operation and write your answer in standard form.

$$\frac{7-i}{2+10i}$$

Hint

Recall that standard form does not allow any i's in the denominator.

Step 1

Because standard form does not allow for i's to be in the denominator we'll need to multiply the numerator and denominator by the conjugate of the denominator, which is $2-10i$.

Step 2

Multiplying by the conjugate gives,

$$\frac{7-i}{2+10i}\,\frac{2-10i}{2-10i} = \frac{(7-i)\,(2-10i)}{(2+10i)\,(2-10i)}$$

Step 3

Now all we need to do is do the multiplication in the numerator and denominator and put the result in standard form.

$$\frac{7-i}{2+10i} = \frac{14-72i+10i^2}{4-100i^2} = \frac{4-72i}{104} = \frac{4}{104} - \frac{72}{104}i = \boxed{\frac{1}{26} - \frac{9}{13}i}$$

8. Perform the indicated operation and write your answer in standard form.

$$\frac{1+5i}{-3i}$$

Hint

Recall that standard form does not allow any i's in the denominator.

Step 1

Because standard form does not allow for i's to be in the denominator we'll need to multiply the numerator and denominator by the conjugate of the denominator, which is $3i$.

Step 2

Multiplying by the conjugate gives,

$$\frac{1+5i}{-3i}\,\frac{3i}{3i} = \frac{(1+5i)\,(3i)}{(-3i)\,(3i)}$$

Step 3

Now all we need to do is do the multiplication in the numerator and denominator and put the result in standard form.

$$\frac{1+5i}{-3i} = \frac{3i+15i^2}{-9i^2} = \frac{-15+3i}{9} = -\frac{15}{9}+\frac{3}{9}i = \boxed{-\frac{5}{3}+\frac{1}{3}i}$$

---

9. Perform the indicated operation and write your answer in standard form.

$$\frac{6+7i}{8-i}$$

Hint

Recall that standard form does not allow any i's in the denominator.

Step 1

Because standard form does not allow for i's to be in the denominator we'll need to multiply the numerator and denominator by the conjugate of the denominator, which is $8+i$.

Step 2

Multiplying by the conjugate gives,

$$\frac{6+7i}{8-i}\,\frac{8+i}{8+i} = \frac{(6+7i)(8+i)}{(8-i)(8+i)}$$

Step 3

Now all we need to do is do the multiplication in the numerator and denominator and put the result in standard form.

$$\frac{6+7i}{8-i}\,\frac{8+i}{8+i} = \frac{48+62i+7i^2}{64-i^2} = \frac{41+62i}{65} = \boxed{\frac{41}{65}+\frac{62}{65}i}$$

---

2 Solving Equations and Inequalities

This chapter can, in many ways, be considered the heart of a College Algebra class. The ability to solve equations and inequalities is very important and will be used time and again both in this class and in later classes.

The majority of the chapter will be spent discussing how to solve linear and quadratic equations. We will also look at a few applications of linear and quadratic equations. As we will eventually see the ability to solve quadratic equations will arise in other topics in this section. Some equations can, for example, be reduced to a quadratic equation and when solving equations involving roots we will often end up solving a quadratic equation in the solution process.

In addition we'll solve inequalities involving linear equations, polynomial and rational expressions. In these sections we'll again see that the ability to solve linear and quadratic equations key to being able to successfully solve inequalities.

We'll the close out the section by solving equations and inequalities that involve absolute value equations.

The following sections are the practice problems, with solutions, for this material.

2.1 Solution and Solution Sets

1. Is $x = 6$ a solution to $2x - 5 = 3(1 - x) + 22$?

Solution

There really isn't all that much to do for these kinds of problems. All we need to do is plug the given number into both sides of the equation and check to see if the right and left side are the same value.

Here is that work for this particular problem.

$$\begin{aligned} 2(6) - 5 &\stackrel{?}{=} 3(1 - 6) + 22 \\ 7 &= 7 \quad \text{OK} \end{aligned}$$

So, we can see that the right and left sides are the same and so we know that $x = 6$ is a solution to the equation.

2. Is $t = 7$ a solution to $t^2 + 3t - 10 = 4 + 8t$?

Solution

There really isn't all that much to do for these kinds of problems. All we need to do is plug the given number into both sides of the equation and check to see if the right and left side are the same value.

Here is that work for this particular problem.

$$\begin{aligned} (7)^2 + 3(7) - 10 &\stackrel{?}{=} 4 + 8(7) \\ 60 &= 60 \quad \text{OK} \end{aligned}$$

So, we can see that the right and left sides are the same and so we know that $t = 7$ is a solution to the equation.

3. Is $t = -3$ a solution to $t^2 + 3t - 10 = 4 + 8t$?

Solution

There really isn't all that much to do for these kinds of problems. All we need to do is plug the given number into both sides of the equation and check to see if the right and left side are the same value.

Here is that work for this particular problem.

$$\begin{aligned}(-3)^2 + 3(-3) - 10 &\stackrel{?}{=} 4 + 8(-3)\\ -10 &= -20 \quad \text{NOT OK}\end{aligned}$$

So, we can see that the right and left sides are the not the same and so we know that $t = -3$ is not a solution to the equation.

4. Is $w = -2$ a solution to $\dfrac{w^2 + 8w + 12}{w + 2} = 0$?

Solution

There really isn't all that much to do for these kinds of problems. All we need to do is plug the given number into both sides of the equation and check to see if the right and left side are the same value.

Note that for this problem we don't even really need to plug the value into the equation. We can see by a quick inspection that if we were to plug $w = -2$ into this equation we would have division by zero and we know that is not allowed.

Therefore, $w = -2$ is not a solution to this equation.

Be very careful with this kind of problem. If we had plugged $w = -2$ into the equation we'd have gotten zero in the numerator as well and we might be tempted to say that it is a solution to the equation. We'd be wrong however. Regardless of the value of the numerator, we would still have division by zero and that is just not allowed and so $w = -2$ will not be a solution to this equation.

5. Is $z = 4$ a solution to $6z - z^2 \ge z^2 + 3$?

Solution

There really isn't all that much to do for these kinds of problems. All we need to do is plug the given number into both sides of the inequality and check to see if the inequality is true. In this case that will mean checking to see if the left side is larger than or equal to the right side.

Here is that work for this particular problem.

$$\begin{aligned} 6\left(4\right) - \left(4\right)^2 &\overset{?}{\ge} \left(4\right)^2 + 3 \\ 8 &\not\ge 19 \quad \text{NOT OK} \end{aligned}$$

So, we can see that the left side is neither larger than nor equal to the right side and so $z = 4$ is not a solution to this inequality.

6. Is $y = 0$ a solution to $2\left(y + 7\right) - 1 < 4\left(y + 1\right) + 3\left(4y + 10\right)$?

Solution

There really isn't all that much to do for these kinds of problems. All we need to do is plug the given number into both sides of the inequality and check to see if the inequality is true. In this case that will mean checking to see if the left side is less than the right side.

Here is that work for this particular problem.

$$\begin{aligned} 2\left(0 + 7\right) - 1 &\overset{?}{<} 4\left(0 + 1\right) + 3\left(4\left(0\right) + 10\right) \\ 13 &< 34 \quad \text{OK} \end{aligned}$$

So, we can see that the left side is less than the right side and so $y = 0$ is a solution to this inequality.

7. Is $x = 1$ a solution to $(x+1)^2 > 3x+1$?

Solution

There really isn't all that much to do for these kinds of problems. All we need to do is plug the given number into both sides of the inequality and check to see if the inequality is true. In this case that will mean checking to see if the left side is greater than the right side.

Here is that work for this particular problem.

$$\begin{aligned}(1+1)^2 &\overset{?}{>} 3(1)+1\\ 4 &\not> 4 \quad \text{NOT OK}\end{aligned}$$

Be very careful with this type of problem! Four is not greater than 4 (it's equal to 4 - big difference here) and so, we can see that $x = 1$ is not a solution to this inequality.

Contrast the inequality in this problem with,

$$(x+1)^2 \geq 3x+1$$

While $x = 1$ is not a solution to the inequality in the problem statement it is a solution to this inequality since 4 is in fact greater than **or equal to** 4. The presence of the equal sign in the inequality can make all the difference in the world and we really need to be on the lookout for it. It is easy to miss when it's there and it is easy to sometimes assume it is there when in fact it isn't.

2.2 Linear Equations

1. Solve the following equation and check your answer.

$$4x - 7(2 - x) = 3x + 2$$

Step 1

First, we need to clear out the parenthesis on the left side and then simplify the left side.

$$\begin{aligned} 4x - 7(2 - x) &= 3x + 2 \\ 4x - 14 + 7x &= 3x + 2 \\ 11x - 14 &= 3x + 2 \end{aligned}$$

Step 2

Now we can subtract $3x$ and add 14 to both sides to get all the x's on one side and the terms without an x on the other side.

$$\begin{aligned} 11x - 14 &= 3x + 2 \\ 8x &= 16 \end{aligned}$$

Step 3

Finally, all we need to do is divide both sides by the coefficient of the x (*i.e.* the 8) to get the solution of $x = 2$.

Step 4

Now all we need to do is check our answer from Step 3 and verify that it is a solution to the equation. It is important when doing this step to verify by plugging the solution from Step 3 into the equation given in the problem statement.

Here is the verification work.

$$\begin{aligned} 4(2) - 7(2 - 2) &\stackrel{?}{=} 3(2) + 2 \\ 8 &= 8 \qquad \text{OK} \end{aligned}$$

So, we can see that our solution from Step 3 is in fact the solution to the equation.

2. Solve the following equation and check your answer.

$$2(w+3)-10=6(32-3w)$$

Step 1

First, we need to clear out the parenthesis on each side and then simplify each side.

$$\begin{aligned} 2(w+3)-10&=6(32-3w)\\ 2w+6-10&=192-18w\\ 2w-4&=192-18w \end{aligned}$$

Step 2

Now we can add $18w$ and 4 to both sides to get all the w's on one side and the terms without an w on the other side.

$$\begin{aligned} 2w-4&=192-18w\\ 20w&=196 \end{aligned}$$

Step 3

Finally, all we need to do is divide both sides by the coefficient of the w (*i.e.* the 20) to get the solution of $w=\frac{196}{20}=\frac{49}{5}$.

Don't get excited about solutions that are fractions. They happen more often than people tend to realize.

Step 4

Now all we need to do is check our answer from Step 3 and verify that it is a solution to the equation. It is important when doing this step to verify by plugging the solution from Step 3 into the equation given in the problem statement.

Here is the verification work.

$$\begin{aligned} 2\left(\frac{49}{5}+3\right)-10&\stackrel{?}{=}6\left(32-3\left(\frac{49}{5}\right)\right)\\ 2\left(\frac{64}{5}\right)-10&\stackrel{?}{=}6\left(\frac{13}{5}\right)\\ \frac{78}{5}&=\frac{78}{5} \qquad \text{OK} \end{aligned}$$

So, we can see that our solution from Step 3 is in fact the solution to the equation.

3. Solve the following equation and check your answer.

$$\frac{4-2z}{3} = \frac{3}{4} - \frac{5z}{6}$$

Step 1

The first step here is to multiply both sides by the LCD, which happens to be 12 for this problem.

$$\begin{aligned} 12\left(\frac{4-2z}{3}\right) &= 12\left(\frac{3}{4} - \frac{5z}{6}\right) \\ 12\left(\frac{4-2z}{3}\right) &= 12\left(\frac{3}{4}\right) - 12\left(\frac{5z}{6}\right) \\ 4\left(4-2z\right) &= 3\left(3\right) - 2\left(5z\right) \end{aligned}$$

Step 2

Now we need to find the solution and so all we need to do is go through the same process that we used in the first two practice problems. Here is that work.

$$\begin{aligned} 4\left(4-2z\right) &= 3\left(3\right) - 2\left(5z\right) \\ 16 - 8z &= 9 - 10z \\ 2z &= -7 \\ z &= -\frac{7}{2} \end{aligned}$$

Step 3

Now all we need to do is check our answer from Step 2 and verify that it is a solution to the equation. It is important when doing this step to verify by plugging the solution from Step 2 into the equation given in the problem statement.

Here is the verification work.

$$\begin{aligned}\frac{4-2\left(-\frac{7}{2}\right)}{3} &\stackrel{?}{=} \frac{3}{4}-\frac{5\left(-\frac{7}{2}\right)}{6}\\ \frac{4+7}{3} &\stackrel{?}{=} \frac{3}{4}-\frac{-\frac{35}{2}}{6}\\ \frac{11}{3} &\stackrel{?}{=} \frac{3}{4}+\frac{35}{12}\\ \frac{11}{3} &= \frac{11}{3} \qquad \text{OK}\end{aligned}$$

So, we can see that our solution from Step 2 is in fact the solution to the equation.

Note that the verification work can often be quite messy so don't get excited about it when it does. Verification is an important step to always remember for these kinds of problems. You should always know if you got the answer correct before you check the answers and/or your instructor grades the problem!

4. Solve the following equation and check your answer.

$$\frac{4t}{t^2-25}=\frac{1}{5-t}$$

Hint

Do not forget to watch out for values of t that we'll need to avoid!

Step 1

Let's first factor the denominator on the left side so we can identify the LCD. While we are at it we will also factor a minus out of the denominator on the right side.

$$\begin{aligned}\frac{4t}{t^2-25} &= \frac{1}{5-t}\\ \frac{4t}{(t-5)(t+5)} &= \frac{1}{-(t-5)}\\ \frac{4t}{(t-5)(t+5)} &= -\frac{1}{t-5}\end{aligned}$$

So, after factoring the left side and factoring the minus sign out of the denominator on the right side we can quickly see that the LCD for this equation is,

$$(t-5)(t+5)$$

From this we can also see that we'll need to avoid $t=5$ and $t=-5$. Remember that we have to avoid division by zero and we will clearly get division by zero with each of these values of t.

Step 2

Next, we need to do find the solution. To get the solution we'll need to multiply both sides by the LCD and the go through the same process we used in the first couple of practice problems. Here is that work.

$$\begin{aligned} (t-5)(t+5)\left(\frac{4t}{(t-5)(t+5)}\right) &= -\left(\frac{1}{t-5}\right)(t-5)(t+5) \\ 4t &= -(t+5) \\ 4t &= -t-5 \\ 5t &= -5 \\ t &= -1 \end{aligned}$$

Step 3

Finally, we need to verify that our answer from Step 2 is in fact a solution.

The first thing to note is that it is not one of the values of t that we need to avoid. Having determined that we know that we do have a potential solution (*i.e.* it's not a value of t we need to avoid) all we need to do is plug the solution into the equation given in the problem statement.

Here is the verification work.

$$\begin{aligned} \frac{4(-1)}{(-1)^2-25} &\stackrel{?}{=} \frac{1}{5-(-1)} \\ \frac{-4}{1-25} &\stackrel{?}{=} \frac{1}{5+1} \\ \frac{1}{6} &= \frac{1}{6} \qquad \text{OK} \end{aligned}$$

So, we can see that our solution from Step 2 is in fact the solution to the equation.

5. Solve the following equation and check your answer.

$$\frac{3y+4}{y-1} = 2 + \frac{7}{y-1}$$

Hint

Do not forget to watch out for values of y that we'll need to avoid!

Step 1

First, we can see that the LCD for this equation is,

$$y - 1$$

From this we can also see that we'll need to avoid $y = 1$. Remember that we have to avoid division by zero and we will clearly get division by zero with this value of y.

Step 2

Next, we need to do find the solution. To get the solution we'll need to multiply both sides by the LCD and the go through the same process we used in the first couple of practice problems. Here is that work.

$$\begin{aligned}(y-1)\left(\frac{3y+4}{y-1}\right) &= \left(2+\frac{7}{y-1}\right)(y-1)\\ 3y+4 &= 2(y-1)+7\\ 3y+4 &= 2y+5\\ y &= 1\end{aligned}$$

Step 3

Finally, we need to verify that our answer from Step 2 is in fact a solution and in this case there isn't a lot of work to that process. We can see that our potential solution from Step 2 is in fact the value of y that we need to avoid and so this equation has **no solution**.

We could also see this if we plugged the value of y from Step 2 into the equation given in the problem statement. Had we done that we would have gotten a division by zero in two of the terms! That, of course, is why we needed to avoid $y = 1$.

Note as well that we only have caught the division by zero if we verify by plugging into the equation in the problem statement. Had we checked in the equation we got by multiplying by the LCD it would have appeared to be a solution! This is the reason that we need to always check in the equation from the problem statement.

6. Solve the following equation and check your answer.

$$\frac{5x}{3x-3} + \frac{6}{x+2} = \frac{5}{3}$$

Hint

Do not forget to watch out for values of x that we'll need to avoid!

Step 1

Let's first factor a 3 out of the denominator of the first tem on the left side so we can identify the LCD.

$$\frac{5x}{3\left(x-1\right)}+\frac{6}{x+2}=\frac{5}{3}$$

So, after factoring doing the factoring on the first term we can quickly see that the LCD for this equation is,

$$3\left(x-1\right)\left(x+2\right)$$

From this we can also see that we'll need to avoid $x = 1$ and $x = -2$. Remember that we have to avoid division by zero and we will clearly get division by zero with each of these values of x.

Step 2

Next, we need to do find the solution. To get the solution we'll need to multiply both sides by the LCD and the go through the same process we used in the first couple of practice problems. Here is that work.

$$\begin{aligned}
3\left(x-1\right)\left(x+2\right)\left(\frac{5x}{3\left(x-1\right)}+\frac{6}{x+2}\right) &= \left(\frac{5}{3}\right)\left[3\left(x-1\right)\left(x+2\right)\right]\\
5x\left(x+2\right)+3\left(x-1\right)\left(6\right) &= 5\left(x-1\right)\left(x+2\right)\\
5x^2+10x+18\left(x-1\right) &= 5\left(x^2+x-2\right)\\
5x^2+10x+18x-18 &= 5x^2+5x-10\\
28x-18 &= 5x-10\\
23x &= 8\\
x &= \frac{8}{23}
\end{aligned}$$

Step 3

Finally, we need to verify that our answer from Step 2 is in fact a solution.

The first thing to note is that it is not one of the values of x that we need to avoid. Having determined that we know that we do have a potential solution (*i.e.* it's not a value of x we need to avoid) all we need to do is plug the solution into the equation given in the problem statement.

Here is the verification work.

$$\begin{aligned}
\frac{5\left(\frac{8}{23}\right)}{3\left(\frac{8}{23}\right)-3}+\frac{6}{\left(\frac{8}{23}\right)+2} &\stackrel{?}{=} \frac{5}{3}\\
\frac{\frac{40}{23}}{-\frac{45}{23}}+\frac{6}{\frac{54}{23}} &\stackrel{?}{=} \frac{5}{3}\\
-\frac{8}{9}+\frac{23}{9} &\stackrel{?}{=} \frac{5}{3}\\
\frac{5}{3} &= \frac{5}{3} \qquad \text{OK}
\end{aligned}$$

So, we can see that our solution from Step 2 is in fact the solution to the equation.

With this problem we have seen that both the solution and the verification step can be somewhat “messy”. That will happen on occasion and we shouldn’t get excited about it when it does. It is just the way these problems work on occasion.

2.3 Applications of Linear Equations

1. A widget is being sold in a store for \$135.40 and has been marked up 7%. How much did the store pay for the widget?

Step 1

We'll start by letting p be the price that the store paid for the widget. The widget has been marked up by 7% and so 0.07 p has been added to the original price, p.

The equation we get for this problem is then,

$$\begin{aligned} p + 0.07p &= 135.4 \\ 1.07p &= 135.4 \end{aligned}$$

Step 2

To finish all we need to do is divide both sides by 1.07 to get the price the store paid for the widget.

$$p = 126.5421$$

So, with rounding the store paid \$126.54 for the widget.

2. A store is having a 30% off sale and one item is now being sold for \$9.95. What was the original price of the item?

Step 1

We'll start by letting p be the original price of the item. The price of the item has been reduced by 30% and so 0.30 p has been subtracted from the original price, p.

The equation we get for this problem is then,

$$\begin{aligned} p - 0.3p &= 9.95 \\ 0.7p &= 9.95 \end{aligned}$$

Step 2

To finish all we need to do is divide both sides by 0.7 to get the original price of the item.

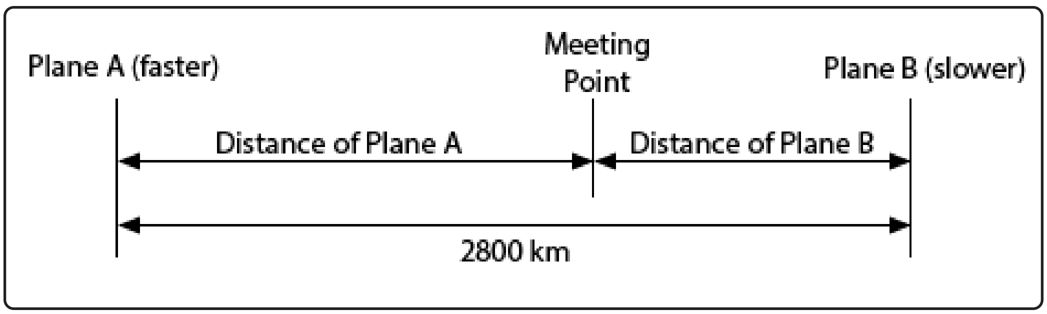
$$p = 14.2143$$

So, with rounding the original price of the item was \$14.21.

3. Two planes start out 2800 km apart and move towards each other meeting after 3.5 hours. One plane flies at 75 km/hour slower than the other plane. What was the speed of each plane?

Step 1

Let's start with a diagram of what is going on in this situation.



Step 2

We can next set up a word equation for this situation.

$$\left(\begin{array}{c}\text{Distance}\\ \text{of Plane A}\end{array}\right)+\left(\begin{array}{c}\text{Distance}\\ \text{of Plane B}\end{array}\right)=2800$$

We know that Distance = Rate X Time so this gives to following word equation.

$$\left(\begin{array}{c}\text{Rate of}\\ \text{Plane A}\end{array}\right)\left(\begin{array}{c}\text{Time of}\\ \text{Plane A}\end{array}\right)+\left(\begin{array}{c}\text{Rate of}\\ \text{Plane B}\end{array}\right)\left(\begin{array}{c}\text{Time of}\\ \text{Plane B}\end{array}\right)=2800$$

Step 3

Let's let r be the speed of the faster plane. Therefore, the speed of the slower plane is $r-75$. We also know that each plane travels for 3.5 hours. Plugging all this information into the word equation above gives the following equation.

$$\begin{aligned}(r)(3.5)+(r-75)(3.5)&=2800\\ 3.5r+3.5(r-75)&=2800\end{aligned}$$

Step 4

Now we can solve this equation for the speed of the faster plane.

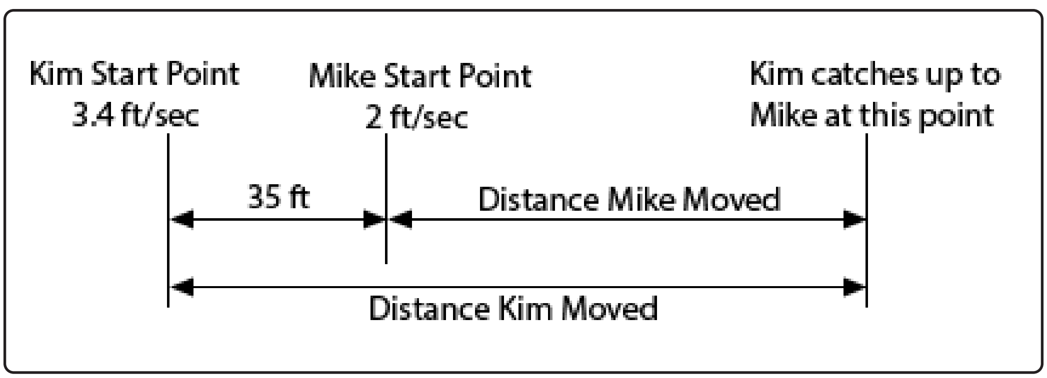
$$\begin{aligned} 3.5r + 3.5\left(r - 75\right) &= 2800 \\ 7r - 262.5 &= 2800 \\ 7r &= 3062.5 \\ r &= 437.5 \end{aligned}$$

So, the faster plane is traveling at 437.5 km/hour while the slower plane is traveling at 362.5 km/hour (75 km/hour slower than faster plane).

4. Mike starts out 35 feet in front of Kim and they both start moving towards the right at the same time. Mike moves at 2 ft/sec while Kim moves at 3.4 ft/sec. How long will it take for Kim to catch up with Mike?

Step 1

Let's start with a diagram of what is going on in this situation.



Step 2

We can next set up a word equation for this situation.

$$\left(\begin{array}{c}\text{Distance}\\ \text{Kim Moved}\end{array}\right) = 35 + \left(\begin{array}{c}\text{Distance}\\ \text{Mike Moved}\end{array}\right)$$

We know that Distance = Rate X Time so this gives to following word equation.

$$\left(\begin{array}{c}\text{Rate of}\\\text{Kim}\end{array}\right)\left(\begin{array}{c}\text{Time of}\\\text{Kim}\end{array}\right)=35+\left(\begin{array}{c}\text{Rate of}\\\text{Mike}\end{array}\right)\left(\begin{array}{c}\text{Time of}\\\text{Mike}\end{array}\right)$$

Step 3

Both Kim and Mike move for the same amount of time so let's call that t. We also know that Kim moves at 3.4 ft/sec while Mike moves at 2 ft/sec. Plugging all this information into the word equation above gives the following equation.

$$\begin{aligned}(3.4)(t)&=35+(2)(t)\\3.4t&=35+2t\end{aligned}$$

Step 4

Now we can solve this equation for the time they traveled.

$$\begin{aligned}3.4t&=35+2t\\1.4t&=35\\t&=25\end{aligned}$$

So, Kim will catch up with Mike after she moves for 25 seconds.

5. A pump can empty a pool in 7 hours and a different pump can empty the same pool in 12 hours. How long does it take for both pumps working together to empty the pool?

Step 1

So, if we consider emptying the pool to be one job we have the following word equation describing both pumps working to empty the pool.

$$\left(\begin{array}{c}\text{Portion of job done}\\ \text{by first pump}\end{array}\right)+\left(\begin{array}{c}\text{Portion of job done}\\ \text{by second pump}\end{array}\right)=1\text{ Job}$$

We know that Portion of Job = Work Rate X Work Time so this gives the following word equation.

$$\left(\begin{array}{c}\text{Work Rate of}\\ \text{first pump}\end{array}\right)\left(\begin{array}{c}\text{Work Time}\\ \text{of first pump}\end{array}\right)+\left(\begin{array}{c}\text{Work Rate of}\\ \text{second pump}\end{array}\right)\left(\begin{array}{c}\text{Work Time}\\ \text{of second pump}\end{array}\right)=1$$

Step 2

We'll need the work rates of each pump and for that we can use the information we have in the problem statement on each pump working individually and the following word equation for each pump doing the job individually.

$$\left(\begin{array}{c}\text{Work Rate}\\ \text{of pump}\end{array}\right)\left(\begin{array}{c}\text{Work Time}\\ \text{of pump}\end{array}\right)=1$$

For the first pump we have,

$$\left(\begin{array}{c}\text{Work Rate}\\ \text{of first pump}\end{array}\right)(7)=1 \qquad \Rightarrow \qquad \text{Work Rate of first pump} = \frac{1}{7}$$

and for the second pump we have,

$$\left(\begin{array}{c}\text{Work Rate}\\ \text{of second pump}\end{array}\right)(12)=1 \qquad \Rightarrow \qquad \text{Work Rate of second pump} = \frac{1}{12}$$

Step 3

Now let t be the amount of time it takes both pumps working together to empty the pool. Using this and the work rates we found in the second step our word equation from the first step becomes,

$$\left(\frac{1}{7}\right)(t) + \left(\frac{1}{12}\right)t = 1$$
$$\frac{19}{84}t = 1$$

Step 4

Now we can solve this for t.

$$\frac{19}{84}t = 1 \qquad \Rightarrow \qquad t = \frac{84}{19} = 4.4211$$

So, it will take both pumps approximately 4.4211 hours to empty the pool if they both work together.

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6. John can paint a house in 28 hours. John and Dave can paint the house in 17 hours working together. How long would it take Dave to paint the house by himself?

Step 1

So, if we consider painting the house to be a single job we have the following word equation if both John and Dave work together to paint the house.

$$\left(\begin{array}{c}\text{Portion of job}\\ \text{done by John}\end{array}\right) + \left(\begin{array}{c}\text{Portion of job}\\ \text{done by Dave}\end{array}\right) = 1 \text{ Job}$$

We know that Portion of Job = Work Rate X Work Time so this gives the following word equation.

$$\left(\begin{array}{c}\text{Work Rate}\\ \text{of John}\end{array}\right)\left(\begin{array}{c}\text{Work Time}\\ \text{of John}\end{array}\right) + \left(\begin{array}{c}\text{Work Rate}\\ \text{of Dave}\end{array}\right)\left(\begin{array}{c}\text{Work Time}\\ \text{of Dave}\end{array}\right) = 1$$

Step 2

We know that John can paint the house in 28 hours so we can use the following equation to determine the John's work rate.

$$\left(\begin{array}{c}\text{Work Rate}\\ \text{of John}\end{array}\right)\left(\begin{array}{c}\text{Work Time}\\ \text{of John}\end{array}\right) = \left(\begin{array}{c}\text{Work Rate}\\ \text{of John}\end{array}\right)(28) = 1 \;\Rightarrow\; \text{Work Rate of John} = \frac{1}{28}$$

Similarly, if we let t be the amount of time it takes Dave to paint the house by himself we have the following relationship between the time and work rate of Dave.

$$\left(\begin{array}{c}\text{Work Rate}\\\text{of Dave}\end{array}\right)\left(\begin{array}{c}\text{Work Time}\\\text{of Dave}\end{array}\right)=\left(\begin{array}{c}\text{Work Rate}\\\text{of Dave}\end{array}\right)(t)=1 \quad\Rightarrow\quad \text{Work Rate of Dave} = \frac{1}{t}$$

Step 3

We can now plug in the information from the second step as well as the fact that it takes John and Dave 17 hours to paint the house by themselves into the word equation from the first step to get,

$$\begin{aligned}\left(\frac{1}{28}\right)(17)+\left(\frac{1}{t}\right)(17)&=1\\ \frac{17}{28}+\frac{17}{t}&=1\end{aligned}$$

Step 4

Now we can solve this for t.

$$\begin{aligned}\frac{17}{t}&=1-\frac{17}{28}\\ \frac{17}{t}&=\frac{11}{28}\\ 17&=\frac{11}{28}t\\ \frac{476}{11}&=t \qquad\Rightarrow\qquad t=43.2727\end{aligned}$$

So, it will take Dave approximately 43.2727 hours to paint the house by himself.

---

7. How much of a 20% acid solution should we add to 20 gallons of a 42% acid solution to get a 35% acid solution?

Step 1

We'll start by letting x be the amount of the 20% solution we'll need. This in turn means that we'll have $x+20$ gallons of the 35% solution once we're done mixing.

The basic word equation is then,

$$\begin{pmatrix}\text{Amount of acid}\\\text{in 20\% solution}\end{pmatrix}+\begin{pmatrix}\text{Amount of acid}\\\text{in 42\% solution}\end{pmatrix}=\begin{pmatrix}\text{Amount of acid}\\\text{in 35\% solution}\end{pmatrix}$$

We know that Amount of Acid in Solution = Percentage of Solution X Volume of Solution. This gives the following word equation.

$$(0.20)\begin{pmatrix}\text{Volume of}\\\text{20\% solution}\end{pmatrix}+(0.42)\begin{pmatrix}\text{Volume of}\\\text{42\% solution}\end{pmatrix}=(0.35)\begin{pmatrix}\text{Volume of}\\\text{35\% solution}\end{pmatrix}$$

Step 2

So, plugging all the known information in gives the following equation that we can solve for x.

$$\begin{aligned}0.2x+(0.42)(20)&=0.35(x+20)\\0.2x+8.4&=0.35x+7\\0.15x&=1.4\\x&=9.33\end{aligned}$$

So, we'll need 9.33 gallons of the 20% acid solution.

8. We need 100 liters of a 25% saline solution and we only have a 14% solution and a 60% solution. How much of each should we mix together to get the 100 liters of the 25% solution?

Step 1

We'll start by letting x be the amount of the 14% solution we'll need. This in turn means that we'll need $100 - x$ gallons of the 60% solution.

The basic word equation is then,

$$\left(\begin{array}{c}\text{Amount of salt}\\ \text{in 14\% solution}\end{array}\right) + \left(\begin{array}{c}\text{Amount of salt}\\ \text{in 60\% solution}\end{array}\right) = \left(\begin{array}{c}\text{Amount of salt}\\ \text{in 25\% solution}\end{array}\right)$$

We know that Amount of Salt in Solution = Percentage of Solution X Volume of Solution. This gives the following word equation.

$$(0.14)\left(\begin{array}{c}\text{Volume of}\\ \text{14\% solution}\end{array}\right) + (0.6)\left(\begin{array}{c}\text{Volume of}\\ \text{60\% solution}\end{array}\right) = (0.25)\left(\begin{array}{c}\text{Volume of}\\ \text{25\% solution}\end{array}\right)$$

Step 2

So, plugging all the known information in gives the following equation that we can solve for x.

$$\begin{aligned} 0.14x + 0.6\left(100 - x\right) &= 0.25\left(100\right) \\ 0.14x + 60 - 0.6x &= 25 \\ -0.46x &= -35 \\ x &= 76.09 \end{aligned}$$

So, we'll need 76.09 liters of the 14% saline solution and 23.91 liters of the 60% saline solution.

9. We want to fence in a field whose length is twice the width and we have 80 feet of fencing material. If we use all the fencing material what would the dimensions of the field be?

Step 1

We'll start by letting x be width of the field and so 2 x will be the length of the field.

Next, we have the following word equation for the length of the fencing material.

$$2\,(\text{Length of Fence}) + 2\,(\text{Width of Fence}) = 80$$

Step 2

So, plugging all the known information in gives the following equation that we can solve for x.

$$\begin{aligned} 2\,(x) + 2\,(2x) &= 80 \\ 6x &= 80 \\ x &= 13.33 \end{aligned}$$

So, the width of the fence will be 13.33 feet while the length will be 26.66 feet.

2.4 Equations With More Than One Variable

1. Solve $E = 3v\left(4 - \frac{2}{r}\right)$ for r.

Step 1

Note that there quite a few solution "paths" that you can take to get the solution to this problem. For this solution let's first distribute the 3 v though the parenthesis.

$$E = 12v - \frac{6v}{r}$$

Step 2

Next, let's clear the denominator out by multiplying both sides by r.

$$Er = 12vr - 6v$$

Step 3

Now let's get all the terms with r on one side and the terms without r on the other side. We'll also factor the r out when we're done as well. Doing this gives,

$$\begin{aligned} Er - 12vr &= -6v \\ (E - 12v)\, r &= -6v \end{aligned}$$

Step 4

Finally, all we need to do is divide by both sides by the coefficient of the r to get,

$$\boxed{r = \frac{-6v}{E - 12v}}$$

Note that depending upon the path you chose for your solution you may have something slightly different for your answer. However, you could do some manipulation of your answer to make it look like mine (or you could manipulate mine to make it look like yours).

2. Solve $Q = \frac{6h}{7s} + 4\,(1 - h)$ for h.

Step 1

Note that there quite a few solution “paths” that you can take to get the solution to this problem. For this solution let’s first clear the denominator out by multiplying both sides by 7 s.

$$\begin{aligned}(Q)(7s) &= 7s\left(\frac{6h}{7s}+4(1-h)\right)\\ 7sQ &= 6h+28s(1-h)\\ 7sQ &= 6h+28s-28sh\end{aligned}$$

We also distributed the 28 s through the parenthesis in anticipation of the next step.

Step 2

Now let’s get all the terms with h on one side and the terms without h on the other side. We’ll also factor the h out when we’re done as well. Doing this gives,

$$\begin{aligned}7sQ-28s &= 6h-28sh\\ 7sQ-28s &= (6-28s)h\end{aligned}$$

Step 3

Finally, all we need to do is divide by both sides by the coefficient of the h to get,

$$\boxed{h=\frac{7sQ-28s}{6-28s}}$$

Note that depending upon the path you chose for your solution you may have something slightly different for your answer. However, you could do some manipulation of your answer to make it look like mine (or you could manipulate mine to make it look like yours).

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3. Solve $Q=\dfrac{6h}{7s}+4(1-h)$ for h.

Step 1

Note that there quite a few solution “paths” that you can take to get the solution to this problem. For this solution let’s first clear the denominator out by multiplying both sides by

7 s.

$$\begin{aligned}(Q)(7s) &= 7s\left(\frac{6h}{7s} + 4(1-h)\right)\\ 7sQ &= 6h + 28s(1-h)\\ 7sQ &= 6h + 28s - 28sh\end{aligned}$$

We also distributed the 28 s through the parenthesis in anticipation of the next step.

Step 2

Now let's get all the terms with h on one side and the terms without h on the other side. We'll also factor the h out when we're done as well. Doing this gives,

$$\begin{aligned}7sQ - 28s &= 6h - 28sh\\ 7sQ - 28s &= (6 - 28s)\,h\end{aligned}$$

Step 3

Finally, all we need to do is divide by both sides by the coefficient of the h to get,

$$\boxed{h = \frac{7sQ - 28s}{6 - 28s}}$$

Note that depending upon the path you chose for your solution you may have something slightly different for your answer. However, you could do some manipulation of your answer to make it look like mine (or you could manipulate mine to make it look like yours).

4. Solve $A - \dfrac{1-2t}{4p} = \dfrac{4+3t}{5p}$ for t.

Step 1

Note that there quite a few solution "paths" that you can take to get the solution to this problem. For this solution let's first clear the denominator out by multiplying both sides by 20 p.

$$\begin{aligned} A - \frac{1-2t}{4p} &= \frac{4+3t}{5p} \\ 20p\left(A - \frac{1-2t}{4p}\right) &= 20p\left(\frac{4+3t}{5p}\right) \\ 20Ap - 5\left(1-2t\right) &= 4\left(4+3t\right) \\ 20Ap - 5 + 10t &= 16 + 12t \end{aligned}$$

We also distributed the constants through the parenthesis in anticipation of the next step.

Step 2

Now let's get all the terms with t on one side and the terms without t on the other side. Doing this gives,

$$20Ap - 21 = 2t$$

Step 3

Finally, all we need to do is divide by both sides by the coefficient of the t to get,

$$\boxed{t = \frac{20Ap - 21}{2}}$$

Note that depending upon the path you chose for your solution you may have something slightly different for your answer. However, you could do some manipulation of your answer to make it look like mine (or you could manipulate mine to make it look like yours).

5. Solve $y = \dfrac{10}{3-7x}$ for x.

Step 1

Note that there quite a few solution "paths" that you can take to get the solution to this problem. For this solution let's first clear the denominator out.

$$\begin{aligned} y\left(3-7x\right) &= 10 \\ 3y - 7xy &= 10 \end{aligned}$$

We also distributed the y through the parenthesis in anticipation of the next step.

Step 2

Now let's get all the terms with x on one side and the terms without x on the other side. Doing this gives,

$$-7xy = 10 - 3y$$

Step 3

Finally, all we need to do is divide by both sides by the coefficient of the x to get,

$$\boxed{x = \frac{10-3y}{-7y} = \frac{3y-10}{7y}}$$

We distributed the minus sign in the denominator into the numerator to reduce the number of minus signs in the answer but doesn't need to be done if you don't want to.

Note that depending upon the path you chose for your solution you may have something slightly different for your answer. However, you could do some manipulation of your answer to make it look like mine (or you could manipulate mine to make it look like yours).

6. Solve $y = \dfrac{3+x}{12-9x}$ for x.

Step 1

Note that there quite a few solution "paths" that you can take to get the solution to this problem. For this solution let's first clear the denominator out.

$$\begin{aligned} y\left(12-9x\right) &= 3+x \\ 12y-9xy &= 3+x \end{aligned}$$

We also distributed the y through the parenthesis in anticipation of the next step.

Step 2

Now let's get all the terms with x on one side and the terms without x on the other side. Doing this gives,

$$\begin{aligned} 12y-3 &= x+9xy \\ 12y-3 &= x\left(1+9y\right) \end{aligned}$$

Step 3

Finally, all we need to do is divide by both sides by the coefficient of the x to get,

$$\boxed{x = \frac{12y-3}{1+9y}}$$

Note that depending upon the path you chose for your solution you may have something slightly different for your answer. However, you could do some manipulation of your answer to make it look like mine (or you could manipulate mine to make it look like yours).

2.5 Quadratic Equations - Part I

1. Solve the following quadratic equation by factoring.

$$u^2 - 5u - 14 = 0$$

Step 1

Not much to this problem. We already have zero on one side of the equation, which we need to proceed with this problem. Therefore, all we need to do is actually factor the quadratic.

$$(u + 2)(u - 7) = 0$$

Step 2

Now all we need to do is use the zero factor property to get,

$$\begin{array}{ccc} u + 2 = 0 & & u - 7 = 0 \\ & \text{OR} & \\ u = -2 & & u = 7 \end{array}$$

Therefore the two solutions are : $\boxed{u = -2 \text{ and } u = 7}$

We'll leave it to you to verify that they really are solutions if you'd like to by plugging them back into the equation.

2. Solve the following quadratic equation by factoring.

$$x^2 + 15x = -50$$

Step 1

The first thing we need to do is get everything on one side of the equation and then factor the quadratic.

$$x^2 + 15x + 50 = 0$$
$$(x + 5)(x + 10) = 0$$

Step 2

Now all we need to do is use the zero factor property to get,

$$x + 5 = 0 \quad \text{OR} \quad x + 10 = 0$$
$$x = -5 \qquad\qquad x = -10$$

Therefore the two solutions are : $\boxed{x = -5 \text{ and } x = -10}$

We'll leave it to you to verify that they really are solutions if you'd like to by plugging them back into the equation.

3. Solve the following quadratic equation by factoring.

$$y^2 = 11y - 28$$

Step 1

The first thing we need to do is get everything on one side of the equation and then factor the quadratic.

$$\begin{aligned} y^2 - 11y + 28 &= 0 \\ (y-4)(y-7) &= 0 \end{aligned}$$

Step 2

Now all we need to do is use the zero factor property to get,

$$y - 4 = 0 \quad \text{OR} \quad y - 7 = 0$$
$$y = 4 \qquad\qquad y = 7$$

Therefore the two solutions are : $\boxed{y = 4 \text{ and } y = 7}$

We'll leave it to you to verify that they really are solutions if you'd like to by plugging them back into the equation.

4. Solve the following quadratic equation by factoring.

$$19x = 7 - 6x^2$$

Step 1

The first thing we need to do is get everything on one side of the equation and then factor the quadratic.

$$\begin{aligned} 6x^2 + 19x - 7 &= 0 \\ (3x-1)(2x+7) &= 0 \end{aligned}$$

Step 2

Now all we need to do is use the zero factor property to get,

$$\begin{aligned} 3x - 1 &= 0 \\ x &= \frac{1}{3} \end{aligned} \qquad \text{OR} \qquad \begin{aligned} 2x + 7 &= 0 \\ x &= -\frac{7}{2} \end{aligned}$$

Therefore the two solutions are : $\boxed{x = \frac{1}{3} \text{ and } x = -\frac{7}{2}}$

We'll leave it to you to verify that they really are solutions if you'd like to by plugging them back into the equation.

5. Solve the following quadratic equation by factoring.

$$6w^2 - w = 5$$

Step 1

The first thing we need to do is get everything on one side of the equation and then factor the quadratic.

$$\begin{aligned} 6w^2 - w - 5 &= 0 \\ (6w+5)(w-1) &= 0 \end{aligned}$$

Step 2

Now all we need to do is use the zero factor property to get,

$$\begin{aligned} 6w + 5 &= 0 \\ w &= -\frac{5}{6} \end{aligned} \qquad \text{OR} \qquad \begin{aligned} w - 1 &= 0 \\ w &= 1 \end{aligned}$$

Therefore the two solutions are : $\boxed{w = -\frac{5}{6} \text{ and } w = 1}$

We'll leave it to you to verify that they really are solutions if you'd like to by plugging them back into the equation.

6. Solve the following quadratic equation by factoring.

$$z^2 - 16z + 61 = 2z - 20$$

Step 1

The first thing we need to do is get everything on one side of the equation and then factor the quadratic.

$$\begin{aligned} z^2 - 18z + 81 &= 0 \\ (z-9)^2 &= 0 \end{aligned}$$

Step 2

From the factored form we can quickly see that the solution is

$$\boxed{z = 9}$$

7. Solve the following quadratic equation by factoring.

$$12x^2 = 25x$$

Step 1

The first thing we need to do is get everything on one side of the equation and then factor the quadratic.

$$\begin{aligned} 12x^2 - 25x &= 0 \\ x\left(12x - 25\right) &= 0 \end{aligned}$$

Make sure that you do not just cancel an x from both sides of the equation!

Step 2

Now all we need to do is use the zero factor property to get,

$$x = 0 \qquad \text{OR} \qquad \begin{aligned} 12x - 25 &= 0 \\ x &= \frac{25}{12} \end{aligned}$$

Therefore the two solutions are : $\boxed{x = 0 \text{ and } x = \frac{25}{12}}$

Note that if we'd canceled an x from both sides of the equation in the first step we would have missed the solution $x = 0$!

8. Use factoring to solve the following equation.

$$x^4 - 2x^3 - 3x^2 = 0$$

Step 1

Do not let the fact that this equation is not a quadratic equation convince you that you can't do it! Note that we can factor an x^2 out of the equation. Doing that gives,

$$x^2\left(x^2 - 2x - 3\right) = 0$$

The quantity in the parenthesis is a quadratic and we can factor it. The full factoring of the equation is then,

$$x^2\left(x - 3\right)\left(x + 1\right) = 0$$

Step 2

Now all we need to do is use the zero factor property to get,

$$x^2 = 0 \quad \text{OR} \quad x - 3 = 0 \quad \text{OR} \quad x + 1 = 0$$
$$x = 0 \qquad\qquad x = 3 \qquad\qquad x = -1$$

Therefore the three solutions are : $\boxed{x = 0,\ x = 3 \text{ and } x = -1}$

9. Use factoring to solve the following equation.

$$t^5 = 9t^3$$

Step 1

Do not let the fact that this equation is not a quadratic equation convince you that you can't do it! Note that we move both terms to one side we can factor a t^3 out of the equation. Doing that gives,

$$\begin{aligned} t^5 - 9t^3 &= 0 \\ t^3\left(t^2 - 9\right) &= 0 \end{aligned}$$

The quantity in the parenthesis is a quadratic and we can factor it. The full factoring of the equation is then,

$$t^3\left(t-3\right)\left(t+3\right) = 0$$

Step 2

Now all we need to do is use the zero factor property to get,

$$t^3 = 0 \quad \text{OR} \quad t - 3 = 0 \quad \text{OR} \quad t + 3 = 0$$
$$t = 0 \qquad\qquad t = 3 \qquad\qquad t = -3$$

Therefore the three solutions are : $\boxed{t = 0,\ t = 3 \text{ and } t = -3}$

10. Use factoring to solve the following equation.

$$\frac{w^2 - 10}{w + 2} + w - 4 = w - 3$$

Step 1

This is an equation containing rational expressions so we know that the first step is to clear out the denominator by multiplying by the LCD, which is $w+2$ in this case. Also, note that we now know that we must avoid $w=-2$ so we do not get division by zero.

Multiplying by the LCD and doing some basic simplification gives,

$$\begin{aligned}(w+2)\left(\frac{w^2-10}{w+2}+w-4\right) &= (w-3)(w+2)\\ w^2-10+(w-4)(w+2) &= (w-3)(w+2)\\ w^2-10+w^2-2w-8 &= w^2-w-6\\ w^2-w-12 &= 0\end{aligned}$$

Step 2

We can now factor the quadratic to get,

$$(w-4)(w+3)=0$$

The zero factor property now tells us,

$$\begin{aligned}w-4&=0\\ w&=4\end{aligned} \qquad \text{OR} \qquad \begin{aligned}w+3&=0\\ w&=-3\end{aligned}$$

Therefore the two solutions are : $\boxed{w=4 \text{ and } w=-3}$.

Note as well that because neither of these are $w=-2$ we know that we won't get division by zero. Do not forget this important part of the solution process for equations involving rational expressions!

11. Use factoring to solve the following equation.

$$\frac{4z}{z+1} + \frac{5}{z} = \frac{6z+5}{z^2+z}$$

Step 1

This is an equation containing rational expressions so we know that the first step is to clear out the denominator by multiplying by the LCD, which is $z(z+1)$ in this case. Also, note that we now know that we must avoid $z = 0$ and $z = -1$ so we do not get division by zero.

Multiplying by the LCD and doing some basic simplification gives,

$$\begin{aligned} z(z+1)\left(\frac{4z}{z+1} + \frac{5}{z}\right) &= \left(\frac{6z+5}{z^2+z}\right) z(z+1) \\ (z)(4z) + 5(z+1) &= 6z+5 \\ 4z^2 + 5z + 5 &= 6z+5 \\ 4z^2 - z &= 0 \end{aligned}$$

Step 2

We can now factor the quadratic to get,

$$z(4z-1) = 0$$

The zero factor property now tells us,

$$z = 0 \qquad \text{OR} \qquad \begin{aligned} 4z - 1 &= 0 \\ z &= \frac{1}{4} \end{aligned}$$

Note that we cannot use the first potential solution since that would give us division by zero!

Therefore the only solution is $\boxed{z = \frac{1}{4}}$.

When dealing with equations that have rational expressions do not forget to verify that you do not get division by zero with any of the potential solutions! As we saw in this case if we had not checked we would have gotten a value of z that seemed to be a solution but in fact was not because of the division by zero issue.

12. Use factoring to solve the following equation.

$$x + 1 = \frac{2x-7}{x+5} - \frac{5x+8}{x+5}$$

Step 1

This is an equation containing rational expressions so we know that the first step is to clear out the denominator by multiplying by the LCD, which is $x + 5$ in this case. Also, note that we now know that we must avoid $x = -5$ so we do not get division by zero.

Multiplying by the LCD and doing some basic simplification gives,

$$\begin{aligned}
(x+5)(x+1) &= \left(\frac{2x-7}{x+5} - \frac{5x+8}{x+5}\right)(x+5) \\
(x+5)(x+1) &= \left(\frac{2x-7}{x+5}\right)(x+5) - \left(\frac{5x+8}{x+5}\right)(x+5) \\
(x+5)(x+1) &= 2x-7-(5x+8) \\
x^2+6x+5 &= 2x-7-5x-8 \\
x^2+9x+20 &= 0
\end{aligned}$$

Step 2

We can now factor the quadratic to get,

$$(x+4)(x+5) = 0$$

The zero factor property now tells us,

$$\begin{aligned} x+4&=0 \\ x&=-4 \end{aligned} \qquad \text{OR} \qquad \begin{aligned} x+5&=0 \\ x&=-5 \end{aligned}$$

Note that we cannot use the second potential solution since that would give us division by zero! Therefore the only solution is : $\boxed{x = -4}$.

When dealing with equations that have rational expressions do not forget to verify that you do not get division by zero with any of the potential solutions! As we saw in this case if we had not checked we would have gotten a value of x that seemed to be a solution but in fact was not because of the division by zero issue.

13. Use the Square Root Property to solve the equation.

$$9u^2 - 16 = 0$$

Step 1

There really isn't too much to this problem. Just recall that we need to get the variable on one side of the equation by itself with a coefficient of one. For this problem that gives,

$$\begin{aligned} 9u^2 &= 16 \\ u^2 &= \frac{16}{9} \end{aligned}$$

Step 2

Now all we need to do is use the Square Root Property to get,

$$u = \pm\sqrt{\frac{16}{9}} = \pm\frac{\sqrt{16}}{\sqrt{9}} = \pm\frac{4}{3}$$

So we have the following two solutions : $\boxed{u = -\frac{4}{3} \text{ and } u = \frac{4}{3}}$.

---

14. Use the Square Root Property to solve the equation.

$$x^2 + 15 = 0$$

Step 1

There really isn't too much to this problem. Just recall that we need to get the variable on one side of the equation by itself with a coefficient of one. For this problem that gives,

$$x^2 = -15$$

Step 2

Now all we need to do is use the Square Root Property to get,

$$x = \pm\sqrt{-15} = \pm\sqrt{15}\,i$$

So we have the following two solutions $\boxed{x = -\sqrt{15}\,i \text{ and } x = \sqrt{15}\,i}$.

Do not get excited about complex solutions. They will happen fairly regularly when solving quadratic equations so we need to be able to deal with them.

15. Use the Square Root Property to solve the equation.

$$(z-2)^2 - 36 = 0$$

Step 1

There really isn't too much to this problem. Just recall that we need to get the squared term on one side of the equation by itself with a coefficient of one. For this problem that gives,

$$(z-2)^2 = 36$$

Step 2

Using the Square Root Property gives,

$$z - 2 = \pm\sqrt{36} = \pm 6$$

To finish this off all we need to do then is solve for z by adding 2 to both sides. This gives,

$$z = 2 \pm 6 \qquad \Rightarrow \qquad z = 2 - 6 = -4, \quad z = 2 + 6 = 8$$

So, after we did a little arithmetic, have the following two solutions : $\boxed{z = -4 \text{ and } z = 8}$.

16. Use the Square Root Property to solve the equation.

$$(6t+1)^2 + 3 = 0$$

Step 1

There really isn't too much to this problem. Just recall that we need to get the squared term on one side of the equation by itself with a coefficient of one. For this problem that gives,

$$(6t+1)^2 = -3$$

Step 2

Using the Square Root Property gives,

$$6t + 1 = \pm\sqrt{-3} = \pm\sqrt{3}\,i$$

To finish this off all we need to do then is solve for t by subtracting 1 from both sides and then dividing by the 6. This gives,

$$\begin{aligned} 6t &= -1 \pm \sqrt{3}\, i \\ t &= \frac{-1 \pm \sqrt{3}\, i}{6} = -\frac{1}{6} \pm \frac{\sqrt{3}}{6} i \end{aligned}$$

Note that we did a little rewrite after dividing by the 6 to put the answer in a more standard form for complex numbers.

We then have the following two solutions : $\boxed{t = -\frac{1}{6} - \frac{\sqrt{3}}{6} i \text{ and } t = -\frac{1}{6} + \frac{\sqrt{3}}{6} i}$.

2.6 Quadratic Equations - Part II

1. Complete the square on the following expression.

$$x^2 + 8x$$

Step 1

First, we need to identify the number we need to add to this. Recall that we will need the coefficient of the x to do this. The number we need is,

$$\left(\frac{8}{2}\right)^2 = (4)^2 = 16$$

Step 2

To complete the square all we need to do then is add this to the expression and factor the result. Doing this gives,

$$\boxed{x^2 + 8x + 16 = (x+4)^2}$$

2. Complete the square on the following expression.

$$u^2 - 11u$$

Step 1

First, we need to identify the number we need to add to this. Recall that we will need the coefficient of the u to do this. The number we need is,

$$\left(\frac{-11}{2}\right)^2 = \frac{(-11)^2}{(2)^2} = \frac{121}{4}$$

Step 2

To complete the square all we need to do then is add this to the expression and factor the result. Doing this gives,

$$\boxed{u^2 - 11u + \frac{121}{4} = \left(u - \frac{11}{2}\right)^2}$$

Recall that this will always factor as u plus the number inside the parenthesis in the first step, $-\frac{11}{2}$ in this case.

Do not get too excited about the fractions that can show up in these problems. They will be there occasionally and so we need to be able to deal with them. Luckily, if you can recall the “trick” to the factoring they aren’t all that bad.

---

3. Complete the square on the following expression.

$$2z^2 - 12z$$

Step 1

Remember that prior to completing the square we need a coefficient of one on the squared variable. However, we can’t just “cancel” it since that requires an equation which we don’t have.

Therefore, we need to first factor a 2 out of the expression as follows,

$$2z^2 - 12z = 2\left(z^2 - 6z\right)$$

We can now proceed with completing the square on the expression inside the parenthesis.

Step 2

Next, we’ll need the number we need to add onto the expression inside the parenthesis. We’ll need the coefficient of the z to do this. The number we need is,

$$\left(\frac{-6}{2}\right)^2 = (-3)^2 = 9$$

Step 3

To complete the square all we need to do then is add this to the expression inside the parenthesis and factor the result. Doing this gives,

$$\boxed{2z^2 - 12z = 2\left(z^2 - 6z + 9\right) = 2(z-3)^2}$$

Be careful when the coefficient of the squared term is not a one! In order to get the correct answer to completing the square we must have a coefficient of one on the squared term!

---

4. Solve the following quadratic equation by completing the square.

$$t^2 - 10t + 34 = 0$$

Step 1

First, let's get the equation put into the form where all the variables are on one side and the number is on the other side. Doing this gives,

$$t^2 - 10t = -34$$

Step 2

We can now complete the square on the expression on the left side of the equation.

The number that we'll need to do this is,

$$\left(\frac{-10}{2}\right)^2 = (-5)^2 = 25$$

Step 3

At this point we need to recall that we have an equation here and what we do to one side of the equation we also need to do the other. In other words, don't forget to add the number from the previous step to both sides of the equation from Step 1.

$$\begin{aligned} t^2 - 10t + 25 &= -34 + 25 \\ (t-5)^2 &= -9 \end{aligned}$$

Step 4

Now all we need to do to finish solving the equation is to use the Square Root Property on the equation from the previous step. Doing this gives,

$$\begin{aligned} t-5 &= \pm\sqrt{-9} = \pm 3i \\ t &= 5 \pm 3i \end{aligned}$$

The two solutions to this equation are then $\boxed{t = 5-3i \text{ and } t = 5+3i}$.

---

5. Solve the following quadratic equation by completing the square.

$$v^2 + 8v - 9 = 0$$

Step 1

First, let's get the equation put into the form where all the variables are on one side and the number is on the other side. Doing this gives,

$$v^2 + 8v = 9$$

Step 2

We can now complete the square on the expression on the left side of the equation.

The number that we'll need to do this is,

$$\left(\frac{8}{2}\right)^2 = (4)^2 = 16$$

Step 3

At this point we need to recall that we have an equation here and what we do to one side of the equation we also need to do the other. In other words, don't forget to add the number from the previous step to both sides of the equation from Step 1.

$$\begin{aligned} v^2 + 8v + 16 &= 9 + 16 \\ (v+4)^2 &= 25 \end{aligned}$$

Step 4

Now all we need to do to finish solving the equation is to use the Square Root Property on the equation from the previous step. Doing this gives,

$$\begin{aligned} v+4 &= \pm\sqrt{25} = \pm 5 \\ v &= -4 \pm 5 \qquad \Rightarrow \qquad v = -4-5 = -9, \quad v = -4+5 = 1 \end{aligned}$$

The two solutions to this equation are then $\boxed{v = -9 \text{ and } v = 1}$.

---

6. Solve the following quadratic equation by completing the square.

$$x^2 + 9x + 16 = 0$$

Step 1

First, let's get the equation put into the form where all the variables are on one side and the number is on the other side. Doing this gives,

$$x^2 + 9x = -16$$

Step 2

We can now complete the square on the expression on the left side of the equation.

The number that we'll need to do this is,

$$\left(\frac{9}{2}\right)^2 = \frac{(9)^2}{(2)^2} = \frac{81}{4}$$

Step 3

At this point we need to recall that we have an equation here and what we do to one side of the equation we also need to do the other. In other words, don't forget to add the number from the previous step to both sides of the equation from Step 1.

$$\begin{aligned} x^2 + 9x + \frac{81}{4} &= -16 + \frac{81}{4} \\ \left(x + \frac{9}{2}\right)^2 &= \frac{17}{4} \end{aligned}$$

Step 4

Now all we need to do to finish solving the equation is to use the Square Root Property on the equation from the previous step. Doing this gives,

$$\begin{aligned} x + \frac{9}{2} &= \pm\sqrt{\frac{17}{4}} = \pm\frac{\sqrt{17}}{\sqrt{4}} = \pm\frac{\sqrt{17}}{2} \\ x &= -\frac{9}{2} \pm \frac{\sqrt{17}}{2} \end{aligned}$$

The two solutions to this equation are then $\boxed{x = -\frac{9}{2} - \frac{\sqrt{17}}{2} \text{ and } x = -\frac{9}{2} + \frac{\sqrt{17}}{2}}$.

Often, we will get "messy" answers when using completing the square to solve equations. This is not something to get too excited about as many applications that involve solving quadratic equations have this kind of solution and so it is something that we just need to be able to deal with.

7. Solve the following quadratic equation by completing the square.

$$4u^2 - 8u + 5 = 0$$

Step 1

First, let's get the equation put into the form where all the variables are on one side, with a coefficient of one on the u^2, and the number is on the other side. Doing this gives,

$$\begin{aligned} u^2 - 2u + \frac{5}{4} &= 0 \\ u^2 - 2u &= -\frac{5}{4} \end{aligned}$$

Step 2

We can now complete the square on the expression on the left side of the equation.

The number that we'll need to do this is,

$$\left(\frac{-2}{2}\right)^2 = (-1)^2 = 1$$

Step 3

At this point we need to recall that we have an equation here and what we do to one side of the equation we also need to do the other. In other words, don't forget to add the number from the previous step to both sides of the equation from Step 1.

$$\begin{aligned} u^2 - 2u + 1 &= -\frac{5}{4} + 1 \\ (u-1)^2 &= -\frac{1}{4} \end{aligned}$$

Step 4

Now all we need to do to finish solving the equation is to use the Square Root Property on the equation from the previous step. Doing this gives,

$$\begin{aligned} u - 1 &= \pm\sqrt{-\frac{1}{4}} = \pm\frac{\sqrt{1}}{\sqrt{4}}i = \pm\frac{1}{2}i \\ u &= 1 \pm \frac{1}{2}i \end{aligned}$$

The two solutions to this equation are then $\boxed{u = 1 - \frac{1}{2}i \text{ and } u = 1 + \frac{1}{2}i}$.

8. Solve the following quadratic equation by completing the square.

$$2x^2 + 5x + 3 = 0$$

Step 1

First, let's get the equation put into the form where all the variables are on one side, with a coefficient of one on the x^2, and the number is on the other side. Doing this gives,

$$\begin{aligned} x^2 + \frac{5}{2}x + \frac{3}{2} &= 0 \\ x^2 + \frac{5}{2}x &= -\frac{3}{2} \end{aligned}$$

Step 2

We can now complete the square on the expression on the left side of the equation.

The number that we'll need to do this is,

$$\left(\frac{\frac{5}{2}}{2}\right)^2 = \left(\frac{5}{4}\right)^2 = \frac{25}{16}$$

Step 3

At this point we need to recall that we have an equation here and what we do to one side of the equation we also need to do the other. In other words, don't forget to add the number from the previous step to both sides of the equation from Step 1.

$$\begin{aligned} x^2 + \frac{5}{2}x + \frac{25}{16} &= -\frac{3}{2} + \frac{25}{16} \\ \left(x + \frac{5}{4}\right)^2 &= \frac{1}{16} \end{aligned}$$

Step 4

Now all we need to do to finish solving the equation is to use the Square Root Property on the equation from the previous step. Doing this gives,

$$\begin{aligned} x + \frac{5}{4} &= \pm\sqrt{\frac{1}{16}} = \pm\frac{1}{4} \\ x &= -\frac{5}{4} \pm \frac{1}{4} \qquad \Rightarrow \qquad x = -\frac{5}{4} - \frac{1}{4} = -\frac{3}{2}, \quad x = -\frac{5}{4} + \frac{1}{4} = -1 \end{aligned}$$

The two solutions to this equation are then $\boxed{x = -\frac{3}{2} \text{ and } x = -1}$.

---

9. Use the quadratic formula to solve the following quadratic equation.

$$x^2 - 6x + 4 = 0$$

Step 1

There really isn't too much to this problem. First, we need to identify the values for the quadratic formula.

$$a = 1 \qquad b = -6 \qquad c = 4$$

Step 2

Plugging these into the quadratic formula gives,

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(4)}}{2(1)} = \frac{6 \pm \sqrt{20}}{2} = \frac{6 \pm 2\sqrt{5}}{2} = 3 \pm \sqrt{5}$$

The two solutions to this equation are then $\boxed{x = 3 - \sqrt{5} \text{ and } x = 3 + \sqrt{5}}$.

---

10. Use the quadratic formula to solve the following quadratic equation.

$$9w^2 - 6w = 101$$

Step 1

First, we need to get the quadratic equation in standard form. This is,

$$9w^2 - 6w - 101 = 0$$

Step 2

Now we need to identify the values for the quadratic formula.

$$a = 9 \qquad b = -6 \qquad c = -101$$

Step 3

Plugging these into the quadratic formula gives,

$$w = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(9)(-101)}}{2(9)} = \frac{6 \pm \sqrt{3672}}{18} = \frac{6 \pm \sqrt{(36)(102)}}{18}$$
$$= \frac{6 \pm 6\sqrt{102}}{18} = \frac{1 \pm \sqrt{102}}{3}$$

The two solutions to this equation are then $\boxed{w = \frac{1}{3} - \frac{\sqrt{102}}{3} \text{ and } w = \frac{1}{3} + \frac{\sqrt{102}}{3}}$.

11. Use the quadratic formula to solve the following quadratic equation.

$$8u^2 + 5u + 70 = 5 - 7u$$

Step 1

First, we need to get the quadratic equation in standard form. This is,

$$8u^2 + 12u + 65 = 0$$

Step 2

Now we need to identify the values for the quadratic formula.

$$a = 8 \qquad b = 12 \qquad c = 65$$

Step 3

Plugging these into the quadratic formula gives,

$$u = \frac{-12 \pm \sqrt{(12)^2 - 4(8)(65)}}{2(8)} = \frac{-12 \pm \sqrt{-1936}}{16} = \frac{-12 \pm 44i}{16} = \frac{-3 \pm 11i}{4}$$

The two solutions to this equation are then $\boxed{u = \frac{-3}{4} - \frac{11}{4}i \text{ and } u = \frac{-3}{4} + \frac{11}{4}i}$.

12. Use the quadratic formula to solve the following quadratic equation.

$$169 - 20t + 4t^2 = 0$$

Step 1

First, we need to get the quadratic equation in standard form. This is,

$$4t^2 - 20t + 169 = 0$$

Note that in this case we just rearranged the terms to have decreasing exponents.

Step 2

Now we need to identify the values for the quadratic formula.

$$a = 4 \qquad b = -20 \qquad c = 169$$

Step 3

Plugging these into the quadratic formula gives,

$$t = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(4)(169)}}{2(4)} = \frac{20 \pm \sqrt{-2304}}{8} = \frac{20 \pm 48i}{8} = \frac{5}{2} \pm 6i$$

The two solutions to this equation are then $\boxed{t = \frac{5}{2} - 6i \text{ and } t = \frac{5}{2} + 6i}$.

13. Use the quadratic formula to solve the following quadratic equation.

$$2z^2 + z - 72 = z^2 - 2z + 58$$

Step 1

First, we need to get the quadratic equation in standard form. This is,

$$z^2 + 3z - 130 = 0$$

In this case we just moved everything from the right side over to the left side.

Step 2

Now we need to identify the values for the quadratic formula.

$$a = 1 \qquad b = 3 \qquad c = -130$$

Step 3

Plugging these into the quadratic formula gives,

$$z = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(-130)}}{2(1)} = \frac{-3 \pm \sqrt{529}}{2} = \frac{-3 \pm 23}{2}$$

In this case because there are no roots or complex numbers we can go further to reduce the solutions to "nicer" values.

$$z = \frac{-3 - 23}{2} = -13, \quad z = \frac{-3 + 23}{2} = 10$$

The two solutions to this equation are then

$$\boxed{z = -13 \text{ and } z = 10}$$

.

As a final comment we can also note that because the solutions were integers we could have also gotten the answer by factoring! The quadratic (once written in standard form) factors as,

$$(z + 13)(z - 10) = 0$$

and this clearly would arrive at the same solutions.

The point of all this is to note that if more than one technique can be used it won't matter which we use. Regardless of the solution technique used you will arrive at the same solutions.

2.7 Quadratic Equations : A Summary

1. Use the discriminant to determine the type of roots for the following equation. Do not find any roots.

$$169x^2 - 182x + 49 = 0$$

Step 1

There really isn't too much to this problem. First, we need to identify the values for computing the discriminant.

$$a = 169 \qquad b = -182 \qquad c = 49$$

Step 2

Plugging these into the formula for the discriminant gives,

$$b^2 - 4ac = (-182)^2 - 4(169)(49) = 0$$

Step 3

The discriminant is zero and so we know that this equation will have a **double root**.

2. Use the discriminant to determine the type of roots for the following equation. Do not find any roots.

$$x^2 + 28x + 61 = 0$$

Step 1

There really isn't too much to this problem. First, we need to identify the values for computing the discriminant.

$$a = 1 \qquad b = 28 \qquad c = 61$$

Step 2

Plugging these into the formula for the discriminant gives,

$$b^2 - 4ac = (28)^2 - 4(1)(61) = 540$$

Step 3

The discriminant is positive and so we know that this equation will have **two real roots**.

3. Use the discriminant to determine the type of roots for the following equation. Do not find any roots.

$$49x^2 - 126x + 102 = 0$$

Step 1

There really isn't too much to this problem. First, we need to identify the values for computing the discriminant.

$$a = 49 \qquad b = -126 \qquad c = 102$$

Step 2

Plugging these into the formula for the discriminant gives,

$$b^2 - 4ac = (-126)^2 - 4(49)(102) = -4116$$

Step 3

The discriminant is negative and so we know that this equation will have **two complex roots**.

4. Use the discriminant to determine the type of roots for the following equation. Do not find any roots.

$$9x^2 + 151 = 0$$

Step 1

There really isn't too much to this problem. First, we need to identify the values for computing the discriminant.

$$a = 9 \qquad b = 0 \qquad c = 151$$

Step 2

Plugging these into the formula for the discriminant gives,

$$b^2 - 4ac = (0)^2 - 4(9)(151) = -5436$$

Step 3

The discriminant is negative and so we know that this equation will have **two complex roots**.

2.8 Applications of Quadratic Equations

1. The width of a rectangle is 1 m less than twice the length. If the area of the rectangle is 100 m^2 what are the dimensions of the rectangle?

Step 1

We'll start by letting L be the length of the rectangle. From the problem statement we now know that the width of the rectangle is 1 m less than twice the length and so must be $2L-1$.

Step 2

We also know that the area of any rectangle is length times width and we are given that the area of this particular rectangle is 100. Therefore, the equation for this problem is,

$$\begin{aligned} A &= (\text{length})\,(\text{width}) \\ 100 &= (L)\,(2L-1) \\ 100 &= 2L^2 - L \end{aligned}$$

Step 3

This is a quadratic equation and we know how to solve that so let's do that. First, we need to get the quadratic equation in standard form.

$$2L^2 - L - 100 = 0$$

We can now use the quadratic formula on this to get,

$$L = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-100)}}{2(2)} = \frac{1 \pm \sqrt{801}}{4}$$

Step 4

Reducing the two values we got in the previous steps to decimals we arrive at the following two solutions to the quadratic equation from Step 2.

$$L = \frac{1-\sqrt{801}}{4} = -6.8255 \qquad\qquad L = \frac{1+\sqrt{801}}{4} = 7.3255$$

We are dealing with a rectangle and so having a negative length doesn't make much sense. Therefore the first solution to the quadratic equation can't be the length of the rectangle.

This means that the length of the rectangle must be 7.3255 m and the width of the rectangle is then $2(7.3255) - 1 = 13.651$ m .

---

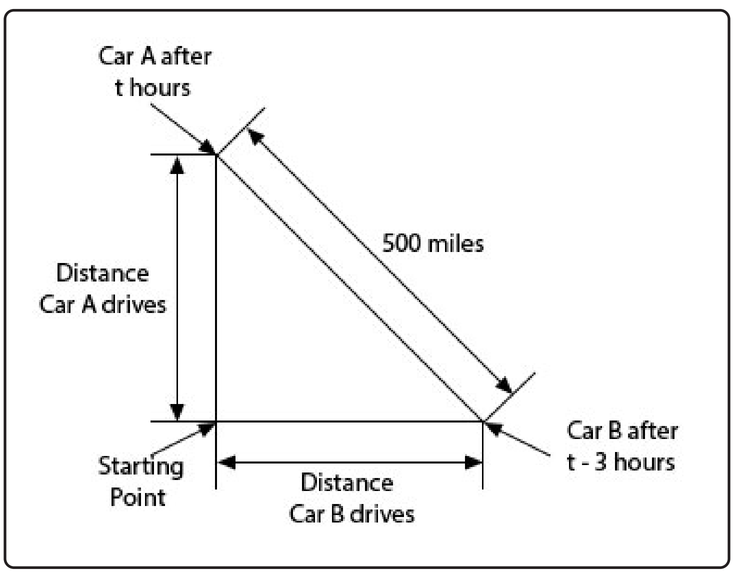
2. Two cars start out at the same spot. One car starts to drive north at 40 mph and 3 hours later the second car starts driving to the east at 60 mph. How long after the first car starts driving does it take for the two cars to be 500 miles apart?

Step 1

Let's start out this problem by defining Car A to be the car that drives 40 mph and Car B to be the car that drives 60 mph. Let's also let t be the time that Car A is driving. From the problem statement we know that Car B starts 3 hours *after* Car A and so drives for 3 hours less than Car A. This means that $t - 3$ is the time that Car B is driving.

Step 2

Next let's set up a sketch for this situation.



Step 3

Okay. Now we need to get an equation for this situation. The first thing to notice about our sketch is that we have a right triangle! This means we can relate all three lengths using the Pythagorean Theorem (this is one of the reasons to have a sketch - to see these kinds of things).

The Pythagorean Theorem tells us that,

$$\left(\begin{array}{c}\text{Distance}\\ \text{Car A drives}\end{array}\right)^2+\left(\begin{array}{c}\text{Distance}\\ \text{Car B drives}\end{array}\right)^2=(500)^2=250,000$$

Step 4

Next, we know that we can find the distance of each car using the formula,

$$\text{Distance}=(\text{Speed of Car})\,(\text{Time driving})$$

So, for each car we have,

$$\begin{aligned}\text{Distance of Car A}&=(40)\,(t)=40t\\ \text{Distance of Car B}&=(60)\,(t-3)=60\,(t-3)\end{aligned}$$

Putting all of this into the "word equation" we wrote down in Step 3 we get the following equation.

$$\begin{aligned}(40t)^2+(60\,(t-3))^2&=250,000\\ 40^2t^2+60^2(t-3)^2&=250,000\\ 1600t^2+3600\left(t^2-6t+9\right)&=250,000\\ 1600t^2+3600t^2-21,600t+32,400&=250,000\\ 5200t^2-21,600t-217,600&=0\end{aligned}$$

Note as well that we did quite a bit of simplification to get the equation into a standard form. Also, do not get excited about the "large" numbers here! They happen on occasion so they are nothing to worry about. This is still just a quadratic and we know how to solve quadratic equations. It doesn't matter if the numbers are single digit numbers of significantly larger numbers as they are here.

Step 5

As noted in the previous step this is just a quadratic equation and we know how to solve those! Using the quadratic formula gives,

$$t = \frac{-(-21,600) \pm \sqrt{(-21,600)^2 - 4(5200)(-217,600)}}{2(5200)} = \frac{21,600 \pm \sqrt{4,992,640,000}}{10,400}$$

Step 6

Reducing the two values we got in the previous steps to decimals we arrive at the following two solutions to the quadratic equation from Step 4.

$$t = \frac{21,600 - \sqrt{4,992,640,000}}{10,400} = -4.7172 \qquad t = \frac{21,600 + \sqrt{4,992,640,000}}{10,400} = 8.8710$$

The first solution to the equation doesn't make any sense since it is negative (we are working with time and so it's safe to assume we are starting at $t = 0$ after all!) so that means the second is the answer we need.

This means that Car A (*i.e.* the one traveling at 40 mph) travels for 8.871 hours while Car B (*i.e.* the one traveling at 60 mph) travels for 5.871 hours (three hours less than Car A time!).

3. Two people can paint a house in 14 hours. Working individually one of the people takes 2 hours more than it takes the other person to paint the house. How long would it take each person working individually to paint the house?

Step 1

First, let Person A be the faster of the two painters and let *t* be the amount of time it takes to paint the house by himself. Next, let Person B be the slower of the two painters and so it will take this person $t + 2$ hours to paint the house by himself.

Step 2

Working together they can paint the house in 14 hours so we have the following word equation for them working together to paint the house.

$$\left(\begin{array}{c}\text{Portion of job}\\ \text{done by Person A}\end{array}\right) + \left(\begin{array}{c}\text{Portion of job}\\ \text{done by Person B}\end{array}\right) = 1 \text{ Job}$$

We know that Portion of Job = Work Rate X Work Time so this gives the following word equation.

$$\left(\begin{array}{c}\text{Work Rate}\\\text{of Person A}\end{array}\right)\left(\begin{array}{c}\text{Work Time}\\\text{of Person A}\end{array}\right)+\left(\begin{array}{c}\text{Work Rate}\\\text{of Person B}\end{array}\right)\left(\begin{array}{c}\text{Work Time}\\\text{of Person B}\end{array}\right)=1$$

$$\left(\begin{array}{c}\text{Work Rate}\\\text{of Person A}\end{array}\right)(14)+\left(\begin{array}{c}\text{Work Rate}\\\text{of Person B}\end{array}\right)(14)=1$$

Step 3

Now we need the work rate of each person which we can get from their individual painting times as follows,

$$\left(\begin{array}{c}\text{Work Rate}\\\text{of Person A}\end{array}\right)\left(\begin{array}{c}\text{Work Time}\\\text{of Person A}\end{array}\right)=\left(\begin{array}{c}\text{Work Rate}\\\text{of Person A}\end{array}\right)(t)=1$$

$$\text{Work Rate of Person A}=\frac{1}{t}$$

$$\left(\begin{array}{c}\text{Work Rate}\\\text{of Person B}\end{array}\right)\left(\begin{array}{c}\text{Work Time}\\\text{of Person B}\end{array}\right)=\left(\begin{array}{c}\text{Work Rate}\\\text{of Person B}\end{array}\right)(t+2)=1$$

$$\text{Work Rate of Person B}=\frac{1}{t+2}$$

Step 4

Plugging these into the word equation from Step 2 we arrive at the following equation.

$$\left(\frac{1}{t}\right)(14)+\left(\frac{1}{t+2}\right)(14)=1$$

$$\frac{14}{t}+\frac{14}{t+2}=1$$

Step 5

To solve this we know that we'll need to multiply by the LCD, $t\,(t+2)$ in this case, to clear the denominators. Doing this gives,

$$\begin{aligned} 14\left(t+2\right)+14t &= t\left(t+2\right) \\ 28t+28 &= t^2+2t \\ t^2-26t-28 &= 0 \end{aligned}$$

After some simplification we arrive a fairly simple quadratic equation to solve. Using the quadratic formula gives,

$$t = \frac{-\left(-26\right) \pm \sqrt{\left(-26\right)^2 - 4\left(1\right)\left(-28\right)}}{2\left(1\right)} = \frac{26 \pm \sqrt{788}}{2}$$

Step 6

Reducing the two values we got in the previous steps to decimals we arrive at the following two solutions to the quadratic equation from Step 2.

$$t = \frac{26-\sqrt{788}}{2} = -1.0357 \qquad t = \frac{26+\sqrt{788}}{2} = 27.0357$$

The first solution to the equation doesn't make any sense since it is negative (we are working with time and so it's safe to assume we are starting at $t = 0$ after all!) so that means the second is the answer we need.

This means that Person A can paint the house in 27.0357 hours while Person B can paint the house in 29.0357 hours (two hours more than Person A).

2.9 Equations Reducible To Quadratic in Form

1. Solve the following equation.

$$x^6 - 9x^3 + 8 = 0$$

Hint

Remember to look at the exponents of the first two terms and try to find a substitution that will turn this into a “normal” quadratic equation.

Step 1

First let’s notice that $6 = 2\left(3\right)$ and so we can use the following substitution to reduce the equation to a quadratic equation.

$$u = x^3 \qquad u^2 = \left(x^3\right)^2 = x^6$$

Step 2

Using this substitution the equation becomes,

$$\begin{aligned} u^2 - 9u + 8 &= 0 \\ \left(u - 1\right)\left(u - 8\right) &= 0 \end{aligned}$$

We can easily see that the solution to this equation is : $u = 1$ and $u = 8$.

Step 3

Now all we need to do is use our substitution from the first step to determine the solution to the original equation.

$$\begin{aligned} u = 1: &\quad x^3 = 1 \quad \Rightarrow \quad x = \left(1\right)^{\frac{1}{3}} = 1 \\ u = 8: &\quad x^3 = 8 \quad \Rightarrow \quad x = \left(8\right)^{\frac{1}{3}} = 2 \end{aligned}$$

Therefore the two solutions to the original equation are : $\boxed{x = 1 \text{ and } x = 2}$.

2. Solve the following equation.

$$x^{-4} - 7x^{-2} - 18 = 0$$

Hint

Remember to look at the exponents of the first two terms and try to find a substitution that will turn this into a "normal" quadratic equation.

Step 1

First let's notice that $-4 = 2(-2)$ and so we can use the following substitution to reduce the equation to a quadratic equation.

$$u = x^{-2} \qquad u^2 = \left(x^{-2}\right)^2 = x^{-4}$$

Step 2

Using this substitution the equation becomes,

$$\begin{aligned} u^2 - 7u - 18 &= 0 \\ (u-9)(u+2) &= 0 \end{aligned}$$

We can easily see that the solution to this equation is : $u = -2$ and $u = 9$.

Step 3

Now all we need to do is use our substitution from the first step to determine the solution to the original equation.

$$u = -2: \quad x^{-2} = \frac{1}{x^2} = -2 \quad \Rightarrow \quad x^2 = -\frac{1}{2} \quad \Rightarrow \quad x = \pm\sqrt{-\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}i$$

$$u = 9: \quad x^{-2} = \frac{1}{x^2} = 9 \quad \Rightarrow \quad x^2 = \frac{1}{9} \quad \Rightarrow \quad x = \pm\sqrt{\frac{1}{9}} = \pm\frac{1}{3}$$

Therefore the four solutions to the original equation are $\boxed{x = \pm\frac{1}{\sqrt{2}}i \text{ and } x = \pm\frac{1}{3}}$.

3. Solve the following equation.

$$4x^{\frac{2}{3}} + 21x^{\frac{1}{3}} + 27 = 0$$

Hint

Remember to look at the exponents of the first two terms and try to find a substitution that will turn this into a "normal" quadratic equation.

Step 1

First let's notice that $\frac{2}{3} = 2\left(\frac{1}{3}\right)$ and so we can use the following substitution to reduce the equation to a quadratic equation.

$$u = x^{\frac{1}{3}} \qquad u^2 = \left(x^{\frac{1}{3}}\right)^2 = x^{\frac{2}{3}}$$

Step 2

Using this substitution the equation becomes,

$$\begin{aligned} 4u^2 + 21u + 27 &= 0 \\ (4u+9)(u+3) &= 0 \end{aligned}$$

We can easily see that the solution to this equation is : $u = -\frac{9}{4}$ and $u = -3$.

Step 3

Now all we need to do is use our substitution from the first step to determine the solution to the original equation.

$$u = -\frac{9}{4}: \qquad x^{\frac{1}{3}} = -\frac{9}{4} \qquad \Rightarrow \qquad x = \left(-\frac{9}{4}\right)^3 = -\frac{729}{64}$$

$$u = -3: \qquad x^{\frac{1}{3}} = -3 \qquad \Rightarrow \qquad x = (-3)^3 = -27$$

Therefore the two solutions to the original equation are $\boxed{x = -\frac{729}{64} \text{ and } x = -27}$.

4. Solve the following equation.

$$x^{10} - 6x^5 + 7 = 0$$

Hint

Remember to look at the exponents of the first two terms and try to find a substitution that will turn this into a "normal" quadratic equation.

Step 1

First let's notice that $10 = 2\left(5\right)$ and so we can use the following substitution to reduce the equation to a quadratic equation.

$$u = x^5 \qquad\qquad u^2 = \left(x^5\right)^2 = x^{10}$$

Step 2

Using this substitution the equation becomes,

$$u^2 - 6u + 7 = 0$$

Now, this equation does not factor. That happens on occasion but luckily enough we know how to solve it anyway. All we need to do is use the quadratic formula to find the solutions to this equation.

$$u = \frac{-\left(-6\right) \pm \sqrt{\left(-6\right)^2 - 4\left(1\right)\left(7\right)}}{2\left(1\right)} = \frac{6 \pm \sqrt{8}}{2} = \frac{6 \pm 2\sqrt{2}}{2} = 3 \pm \sqrt{2}$$

Do not get excited about the "messy" numbers here! These kinds of solutions happen on occasion and we just need to be able to deal with them. Just keep in mind that they are just numbers even if they are not the integers we are used to seeing!

Step 3

Now all we need to do is use our substitution from the first step to determine the solution to the original equation.

$$u = 3 + \sqrt{2}: \qquad x^5 = 3 + \sqrt{2} \qquad \Rightarrow \qquad x = \left(3 + \sqrt{2}\right)^{\frac{1}{5}} = (4.4142)^{\frac{1}{5}} = 1.3458$$

$$u = 3 - \sqrt{2}: \qquad x^5 = 3 - \sqrt{2} \qquad \Rightarrow \qquad x = \left(3 - \sqrt{2}\right)^{\frac{1}{5}} = (1.5858)^{\frac{1}{5}} = 1.0966$$

Therefore the two solutions to the original equation are $\boxed{x = 1.0966 \text{ and } x = 1.3458}$.

---

5. Solve the following equation.

$$\frac{2}{x^2} + \frac{17}{x} + 21 = 0$$

Hint

This works exactly the same as the first four problems even though the x's are in the denominator. The only difference here is that the x's will be in the denominator of our substitution.

Step 1

First let's notice that $2 = 2(1)$ and so we can use the following substitution to reduce the equation to a quadratic equation.

$$u = \frac{1}{x} \qquad u^2 = \left(\frac{1}{x}\right)^2 = \frac{1^2}{x^2} = \frac{1}{x^2}$$

Step 2

Using this substitution the equation becomes,

$$\begin{gathered} 2u^2 + 17u + 21 = 0 \\ (2u + 3)(u + 7) = 0 \end{gathered}$$

We can easily see that the solution to this equation is : $u = -\frac{3}{2}$ and $u = -7$.

Step 3

Now all we need to do is use our substitution from the first step to determine the solution to the original equation.

$$u = -\frac{3}{2}: \qquad \frac{1}{x} = -\frac{3}{2} \qquad \Rightarrow \qquad x = \frac{1}{-\frac{3}{2}} = -\frac{2}{3}$$

$$u = -7: \qquad \frac{1}{x} = -7 \qquad \Rightarrow \qquad x = \frac{1}{-7} = -\frac{1}{7}$$

Therefore the two solutions to the original equation are $\boxed{x = -\frac{2}{3} \text{ and } x = -\frac{1}{7}}$.

---

6. Solve the following equation.

$$\frac{1}{x} - \frac{11}{\sqrt{x}} + 18 = 0$$

Hint

This works exactly the same as the first four problems even though the x's are in the denominator. The only difference here is that the x's will be in the denominator of our substitution.

Step 1

First let's notice that $1 = 2\left(\frac{1}{2}\right)$ and recall that $\sqrt{x} = x^{\frac{1}{2}}$. So we can use the following substitution to reduce the equation to a quadratic equation.

$$u = \frac{1}{\sqrt{x}} \qquad u^2 = \left(\frac{1}{\sqrt{x}}\right)^2 = \left(\frac{1}{x^{\frac{1}{2}}}\right)^2 = \frac{1}{x}$$

Step 2

Using this substitution the equation becomes,

$$\begin{aligned} u^2 - 11u + 18 &= 0 \\ (u-2)(u-9) &= 0 \end{aligned}$$

We can easily see that the solution to this equation is : $u = 2$ and $u = 9$.

Step 3

Now all we need to do is use our substitution from the first step to determine the solution to the original equation.

$$u = 2: \qquad \frac{1}{\sqrt{x}} = 2 \qquad \Rightarrow \qquad \sqrt{x} = \frac{1}{2} \qquad \Rightarrow \qquad x = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$u = 9: \qquad \frac{1}{\sqrt{x}} = 9 \qquad \Rightarrow \qquad \sqrt{x} = \frac{1}{9} \qquad \Rightarrow \qquad x = \left(\frac{1}{9}\right)^2 = \frac{1}{81}$$

Therefore the two solutions to the original equation are $\boxed{x = \frac{1}{4} \text{ and } x = \frac{1}{81}}$.

2.10 Equations with Radicals

1. Solve the following equation.

$$2x = \sqrt{x+3}$$

Step 1

The first step here is to square both sides to get,

$$\begin{aligned}(2x)^2 &= \left(\sqrt{x+3}\right)^2 \\ 4x^2 &= x+3 \\ 4x^2 - x - 3 &= 0\end{aligned}$$

Step 2

This is just a quadratic equation and we know how to solve it so let's do that.

$$(4x+3)(x-1) = 0 \qquad \Rightarrow \qquad x = -\frac{3}{4}, \quad x = 1$$

As shown we have two solutions to the quadratic we got from the first step.

Hint

Recall that the solution process used here can, and often does, introduce values that are not in fact solutions to the original equation!

Step 3

We're not done with this problem. Recall from the notes that the solution process we used here has the unfortunate side effect of sometimes introducing values that are not solutions to the original equation.

So, to finish this out we need to check both of the potential solutions from the previous step in the original equation (recall it's important to check the potential solutions in the original equation).

$$x = -\frac{3}{4}: 2\left(-\frac{3}{4}\right) \stackrel{?}{=} \sqrt{-\frac{3}{4}+3} \quad \to \quad -\frac{3}{2} \stackrel{?}{=} \sqrt{\frac{9}{4}} \quad \to \quad -\frac{3}{2} \neq \frac{3}{2} \quad \text{NOT OK}$$

$$x = 1: \quad 2(1) \stackrel{?}{=} \sqrt{1+3} \quad \to \quad 2 \stackrel{?}{=} \sqrt{4} \quad \to \quad 2 = 2 \quad \text{OK}$$

Only one of the potential solutions work out and so the original equation has a single solution : $\boxed{x = 1}$.

2. Solve the following equation.

$$\sqrt{33 - 2x} = x + 1$$

Step 1

The first step here is to square both sides to get,

$$\begin{aligned} \left(\sqrt{33 - 2x}\right)^2 &= (x+1)^2 \\ 33 - 2x &= x^2 + 2x + 1 \\ x^2 + 4x - 32 &= 0 \end{aligned}$$

Step 2

This is just a quadratic equation and we know how to solve it so let's do that.

$$(x+8)(x-4) = 0 \qquad \Rightarrow \qquad x = -8, \quad x = 4$$

As shown we have two solutions to the quadratic we got from the first step.

Hint

Recall that the solution process used here can, and often does, introduce values that are not in fact solutions to the original equation!

Step 3

We're not done with this problem. Recall from the notes that the solution process we used here has the unfortunate side effect of sometimes introducing values that are not solutions to the original equation.

So, to finish this out we need to check both of the potential solutions from the previous step in the original equation (recall it's important to check the potential solutions in the original equation).

$$x = -8: \qquad \sqrt{33 - 2(-8)} \stackrel{?}{=} -8 + 1 \quad \to \quad \sqrt{49} \stackrel{?}{=} -7 \quad \to \quad 7 \neq -7 \qquad \text{NOT OK}$$

$$x = 4: \qquad \sqrt{33 - 2(4)} \stackrel{?}{=} 4 + 1 \quad \to \quad \sqrt{25} \stackrel{?}{=} 5 \quad \to \quad 5 = 5 \qquad \text{OK}$$

Only one of the potential solutions work out and so the original equation has a single solution : $\boxed{x = 4}$.

3. Solve the following equation.

$$7 = \sqrt{39 + 3x} - x$$

Step 1

The first step here is to square both sides. However, before we do that we need to get the root on one side by itself.

$$7 + x = \sqrt{39 + 3x}$$

Now we can square both sides to get,

$$\begin{aligned} (7 + x)^2 &= \left(\sqrt{39 + 3x}\right)^2 \\ x^2 + 14x + 49 &= 39 + 3x \\ x^2 + 11x + 10 &= 0 \end{aligned}$$

Step 2

This is just a quadratic equation and we know how to solve it so let's do that.

$$(x + 10)(x + 1) = 0 \qquad \Rightarrow \qquad x = -10, \quad x = -1$$

As shown we have two solutions to the quadratic we got from the first step.

Hint

Recall that the solution process used here can, and often does, introduce values that are not in fact solutions to the original equation!

Step 3

We're not done with this problem. Recall from the notes that the solution process we used here has the unfortunate side effect of sometimes introducing values that are not solutions to the original equation.

So, to finish this out we need to check both of the potential solutions from the previous step in the original equation (recall it's important to check the potential solutions in the original equation).

$$x = -10: \quad 7 \stackrel{?}{=} \sqrt{39 + 3(-10)} - (-10) \quad \to \quad 7 \stackrel{?}{=} \sqrt{9} + 10 \quad \to \quad 7 \neq 13 \quad \text{NOT OK}$$

$$x = -1: \quad 7 \stackrel{?}{=} \sqrt{39 + 3(-1)} - (-1) \quad \to \quad 7 \stackrel{?}{=} \sqrt{36} + 1 \quad \to \quad 7 = 7 \quad \text{OK}$$

Only one of the potential solutions work out and so the original equation has a single solution : $\boxed{x = -1}$.

---

4. Solve the following equation.

$$x = 1 + \sqrt{2x - 2}$$

Step 1

The first step here is to square both sides. However, before we do that we need to get the root on one side by itself.

$$x - 1 = \sqrt{2x - 2}$$

Now we can square both sides to get,

$$\begin{aligned} (x-1)^2 &= \left(\sqrt{2x-2}\right)^2 \\ x^2 - 2x + 1 &= 2x - 2 \\ x^2 - 4x + 3 &= 0 \end{aligned}$$

Step 2

This is just a quadratic equation and we know how to solve it so let's do that.

$$(x-1)(x-3) = 0 \qquad \Rightarrow \qquad x = 1, \quad x = 3$$

As shown we have two solutions to the quadratic we got from the first step.

Hint

Recall that the solution process used here can, and often does, introduce values that are not in fact solutions to the original equation!

Step 3

We're not done with this problem. Recall from the notes that the solution process we used here has the unfortunate side effect of sometimes introducing values that are not solutions to the original equation.

So, to finish this out we need to check both of the potential solutions from the previous step in the original equation (recall it's important to check the potential solutions in the original equation).

$$x = 1: \quad 1 \stackrel{?}{=} 1 + \sqrt{2(1) - 2} \quad \to \quad 1 \stackrel{?}{=} 1 + \sqrt{0} \quad \to \quad 1 = 1 \quad \text{OK}$$

$$x = 3: \quad 3 \stackrel{?}{=} 1 + \sqrt{2(3) - 2} \quad \to \quad 3 \stackrel{?}{=} 1 + \sqrt{4} \quad \to \quad 3 = 3 \quad \text{OK}$$

Both of the potential solutions work out and so the original equation has a two solutions : $\boxed{x = 1 \text{ and } x = 3}$.

5. Solve the following equation.

$$1 + \sqrt{1 - x} = \sqrt{2x + 4}$$

Step 1

The first step here is to square both sides to get,

$$\begin{aligned} \left(1 + \sqrt{1 - x}\right)^2 &= \left(\sqrt{2x + 4}\right)^2 \\ 1 + 2\sqrt{1 - x} + \left(\sqrt{1 - x}\right)^2 &= 2x + 4 \\ 1 + 2\sqrt{1 - x} + 1 - x &= 2x + 4 \\ 2\sqrt{1 - x} &= 3x + 2 \end{aligned}$$

Step 2

Unlike the previous problems squaring both sides once isn't sufficient to eliminate the square roots from the problem. So, once we get the remaining root on one side by itself, as we did in the previous step, we need to square both sides once again.

Doing that gives,

$$\begin{aligned} \left(2\sqrt{1 - x}\right)^2 &= (3x + 2)^2 \\ 4(1 - x) &= 9x^2 + 12x + 4 \\ 9x^2 + 16x &= 0 \end{aligned}$$

Step 3

This is just a quadratic equation and we know how to solve it so let's do that.

$$x\left(9x+16\right)=0 \qquad \Rightarrow \qquad x=0, \quad x=-\frac{16}{9}$$

As shown we have two solutions to the quadratic we got from the first step.

Hint

Recall that the solution process used here can, and often does, introduce values that are not in fact solutions to the original equation!

Step 4

We're not done with this problem. Recall from the notes that the solution process we used here has the unfortunate side effect of sometimes introducing values that are not solutions to the original equation.

So, to finish this out we need to check both of the potential solutions from the previous step in the original equation (recall it's important to check the potential solutions in the original equation).

$$x=0: \qquad 1+\sqrt{1-0}\stackrel{?}{=}\sqrt{2\left(0\right)+4} \qquad \to \qquad 1+\sqrt{1}\stackrel{?}{=}\sqrt{4} \qquad \to \qquad 2=2 \qquad \text{OK}$$

$$x=-\frac{16}{9}: \qquad 1+\sqrt{1-\left(-\frac{16}{9}\right)}\stackrel{?}{=}\sqrt{2\left(-\frac{16}{9}\right)+4}$$
$$\to \qquad 1+\sqrt{\frac{25}{9}}\stackrel{?}{=}\sqrt{\frac{4}{9}} \qquad \to \qquad \frac{8}{3}\neq\frac{2}{3} \qquad \text{NOT OK}$$

Only one of the potential solutions work out and so the original equation has a single solution : $\boxed{x=0}$.

2.11 Linear Inequalities

1. Solve the following inequality and give the solution in both inequality and interval notation.

$$4(z+2)-1>5-7(4-z)$$

Hint

Remember that solving linear inequalities is pretty much the same as solving a linear equation. Just remember to be careful when multiplying/dividing by a negative number.

Step 1

We know that the process of solving a linear inequality is pretty much the same process as solving a linear equation. We do basic algebraic manipulations to get all the z's on one side of the inequality and the numbers on the other side. Just remember that what you do to one side of the inequality you have to do to the other side as well. So, let's go through the solution process for this linear inequality.

First, we should clear out the parenthesis on both sides and do any simplification that we can. Doing this gives,

$$\begin{aligned}4z+8-1&>5-28+7z\\4z+7&>-23+7z\end{aligned}$$

Step 2

We can now subtract $7z$ from both sides and subtract 7 to both sides to get,

$$-3z>-30$$

Note that we could just have easily subtracted 4 z from both sides and added 23 to both sides. Each will get the same result in the end.

Step 3

For the final step we need to divide both sides by -3. Recall however that because we are dividing by a negative number we need to switch the direction of the inequality to get,

$$z<10$$

So, the inequality form of the solution is $\boxed{z<10}$ and the interval notation form of the solution is $\boxed{(-\infty, 10)}$.

Remember that we use a parenthesis, *i.e.* ")", for the right side of the interval notation because we are not including 10 in the solution. Also recall that infinities always get parenthesis!

2. Solve the following inequality and give the solution in both inequality and interval notation.

$$\frac{1}{2}(3+4t) \le 6\left(\frac{1}{3}-\frac{1}{2}t\right) - \frac{1}{4}(2+10t)$$

Hint

Remember that solving linear inequalities is pretty much the same as solving a linear equation. Just remember to be careful when multiplying/dividing by a negative number.

Step 1

We know that the process of solving a linear inequality is pretty much the same process as solving a linear equation. We do basic algebraic manipulations to get all the t's on one side of the inequality and the numbers on the other side. Just remember that what you do to one side of the inequality you have to do to the other side as well. So, let's go through the solution process for this linear inequality.

First, we should clear out the parenthesis on both sides and do any simplification that we can. Doing this gives,

$$\frac{3}{2} + 2t \le 2 - 3t - \frac{1}{2} - \frac{5}{2}t$$
$$\frac{3}{2} + 2t \le \frac{3}{2} - \frac{11}{2}t$$

Step 2

We can now add $\frac{11}{2}t$ to both sides and subtract $\frac{3}{2}$ from both sides to get,

$$\frac{15}{2}t \le 0$$

Step 3

For the final step we need to multiply both sides by $\frac{2}{15}$ to get,

$$t \le 0$$

So, the inequality form of the solution is $\boxed{t \le 0}$ and the interval notation form of the solution is $\boxed{(-\infty, 0]}$.

Remember that we use a square bracket, *i.e.* "]", for the right portion of the interval because we are including zero in the solution. Also recall that infinities never get square brackets!

3. Solve the following inequality and give the solution in both inequality and interval notation.

$$-1 < 4x + 2 < 10$$

Hint

Solving double inequalities uses the same basic process as solving single inequalities. Just remember that what you do to one part you have to do to all parts of the inequality.

Step 1

Just like with single inequalities solving these follow pretty much the same process as solving a linear equation. The only difference between this and a single inequality is that we now have three parts of the inequality and so we just need to remember that what we do to one part we need to do to all parts.

Also, recall that the main goal is to get the variable all by itself in the middle and all the numbers on the two outer parts of the inequality.

So, let's start by subtracting 2 from all the parts. This gives,

$$-3 < 4x < 8$$

Step 2

Finally, all we need to do is divide all three parts by 4 to get,

$$-\frac{3}{4} < x < 2$$

So, the inequality form of the solution is $\boxed{-\frac{3}{4} < x < 2}$ and the interval notation form of the solution is $\boxed{\left(-\frac{3}{4}, 2\right)}$.

4. Solve the following inequality and give the solution in both inequality and interval notation.

$$8 \le 3 - 5z < 12$$

Hint

Solving double inequalities uses the same basic process as solving single inequalities. Just remember that what you do to one part you have to do to all parts of the inequality.

Step 1

Just like with single inequalities solving these follow pretty much the same process as solving a linear equation. The only difference between this and a single inequality is that we now have three parts of the inequality and so we just need to remember that what we do to one part we need to do to all parts.

Also, recall that the main goal is to get the variable all by itself in the middle and all the numbers on the two outer parts of the inequality.

So, let's start by subtracting 3 from all the parts. This gives,

$$5 \le -5z < 9$$

Step 2

Finally, all we need to do is divide all three parts by -5 to get,

$$-1 \ge z > -\frac{9}{5}$$

Don't forget that because we were dividing everything by a negative number we needed to switch the direction of the inequalities.

So, the inequality form of the solution is $\boxed{-\frac{9}{5} < z \le -1}$ (we flipped the inequality around to get the smaller number on the left as that is a more "standard" form). The interval notation form of the solution is $\boxed{\left(-\frac{9}{5}, -1\right]}$.

For the interval notation form remember that the smaller number is always on the left (hence the reason for flipping the inequality form above!) and be careful with parenthesis and square brackets. We use parenthesis if we don't include the number and square brackets if we do include the number.

5. Solve the following inequality and give the solution in both inequality and interval notation.

$$0 \le 10w - 15 \le 23$$

Hint

Solving double inequalities uses the same basic process as solving single inequalities. Just remember that what you do to one part you have to do to all parts of the inequality.

Step 1

Just like with single inequalities solving these follow pretty much the same process as solving a linear equation. The only difference between this and a single inequality is that we now have three parts of the inequality and so we just need to remember that what we do to one part we need to do to all parts.

Also, recall that the main goal is to get the variable all by itself in the middle and all the numbers on the two outer parts of the inequality.

So, let's start by add 15 to all the parts. This gives,

$$15 \le 10w \le 38$$

Step 2

Finally, all we need to do is divide all three parts by 10 to get,

$$\frac{3}{2} \le w \le \frac{19}{5}$$

So, the inequality form of the solution is $\boxed{\frac{3}{2} \le w \le \frac{19}{5}}$ and the interval notation form of the solution is $\boxed{\left[\frac{3}{2}, \frac{19}{5}\right]}$.

---

6. Solve the following inequality and give the solution in both inequality and interval notation.

$$2 < \frac{1}{6} - \frac{1}{2}x \le 4$$

Hint

Solving double inequalities uses the same basic process as solving single inequalities. Just remember that what you do to one part you have to do to all parts of the inequality.

Step 1

Just like with single inequalities solving these follow pretty much the same process as solving a linear equation. The only difference between this and a single inequality is that we now have three parts of the inequality and so we just need to remember that what we do to one part we need to do to all parts.

Also, recall that the main goal is to get the variable all by itself in the middle and all the numbers on the two outer parts of the inequality.

So, let's start by subtracting $\frac{1}{6}$ from all the parts. This gives,

$$\frac{11}{6} < -\frac{1}{2}x \leq \frac{23}{6}$$

Step 2

Finally, all we need to do is multiply all three parts by -2 to get,

$$-\frac{11}{3} > x \geq -\frac{23}{3}$$

Don't forget that because we were multiplying everything by a negative number we needed to switch the direction of the inequalities.

So, the inequality form of the solution is $\boxed{-\frac{23}{3} \leq x < -\frac{11}{3}}$ (we flipped the inequality around to get the smaller number on the left as that is a more "standard" form). The interval notation form of the solution is $\boxed{\left[-\frac{23}{3}, -\frac{11}{3}\right)}$.

For the interval notation form remember that the smaller number is always on the left (hence the reason for flipping the inequality form above!) and be careful with parenthesis and square brackets. We use parenthesis if we don't include the number and square brackets if we do include the number.

---

7. If $0 \leq x < 3$ determine a and b for the inequality : $a \leq 4x + 1 < b$

Hint

Can you make the middle part of the first inequality look like the middle part of the second inequality?

Step 1

This problem is really the reverse of the previous problems in this section. In the previous problems we started with something like the second inequality (of course we also had numbers in the two outer portions instead of a and b) and we had to manipulate it to get the x by itself in the middle.

The process here is basically the same just in reverse. We need to do algebraic manipulations to make the middle part of the first inequality look like the middle part of the second manipulation. The only real difference is that with the solving problems we added/subtracted the number before we dealt with the coefficient of the x. Here we need to get the coefficient on the x before we get the number.

So, the first thing we'll do is multiply all three parts of the first inequality by 4. This gives,

$$0\left(4\right) \le 4x < 3\left(4\right) \qquad \Rightarrow \qquad 0 \le 4x < 12$$

Step 2

Now all we need to do is add one to all three parts.

$$1 \le 4x + 1 < 13$$

Step 3

Comparing this inequality in the second step to the second inequality in the problem statement we can see that we must have $\boxed{a = 1}$ and $\boxed{b = 13}$.

2.12 Polynomial Inequalities

1. Solve the following inequality.

$$u^2 + 4u \geq 21$$

Step 1

The first thing we need to do is get a zero on one side of the inequality and then, if possible, factor the polynomial.

$$\begin{aligned} u^2 + 4u - 21 &\geq 0 \\ (u+7)(u-3) &\geq 0 \end{aligned}$$

Hint

Where are the only places where the polynomial might change signs?

Step 2

Despite the fact that this is an inequality we first need to know where the polynomial is zero. From the factored from we can quickly see that the polynomial will be zero at,

$$u = -7 \qquad u = 3$$

Remember that these points are important because they are the only places where the polynomial on the left side of the inequality might change sign. Given that we want to know where the polynomial is zero (which we now know) or positive knowing where the polynomial might change sign will help considerably with determining the answer we're looking for.

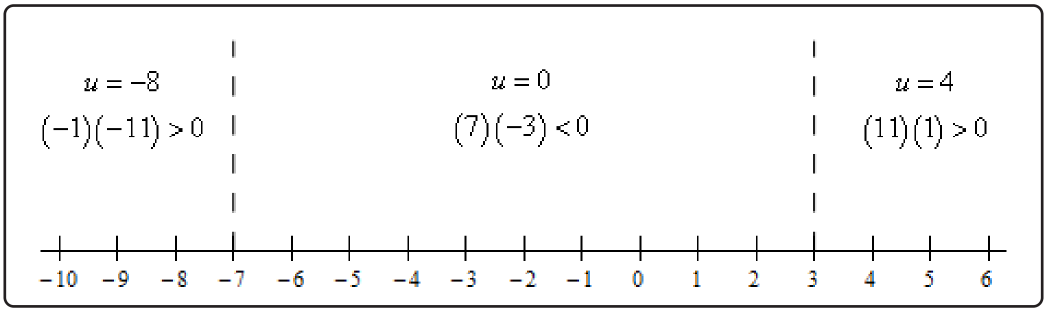
Hint

Knowing that the polynomial can only change sign at the points above how can we quickly determine if the polynomial is positive or negative in the ranges between those points?

Step 3

Recall from the discussion in the notes for this section that because the points from the previous step are the only places where the polynomial might change sign we can quickly determine if the polynomial is positive/negative in the ranges between each of these points simply by plugging in "test points" from each region into the polynomial to check the sign.

So, let's sketch a quick number line with the points where the polynomial is zero graphed on it. We'll also show the test point computations on the number line as well. Here is the number line.



Step 4

All we need to do now is get the solution from the number line in the previous step. Here is both the inequality and interval notation from of the answer.

$$u \le -7 \quad \text{and} \quad u \ge 3$$
$$(-\infty, -7] \quad \text{and} \quad [3, \infty)$$

2. Solve the following inequality.

$$x^2 + 8x + 12 < 0$$

Step 1

The first thing we need to do is get a zero on one side of the inequality (which is already done for this problem) and then, if possible, factor the polynomial.

$$(x+6)(x+2) < 0$$

Hint

Where are the only places where the polynomial might change signs?

Step 2

Despite the fact that this is an inequality we first need to know where the polynomial is zero. From the factored from we can quickly see that the polynomial will be zero at,

$$x = -6 \qquad x = -2$$

Remember that these points are important because they are the only places where the polynomial on the left side of the inequality might change sign. Given that we want to know where the polynomial is negative knowing where the polynomial might change sign will help considerably with determining the answer we're looking for.

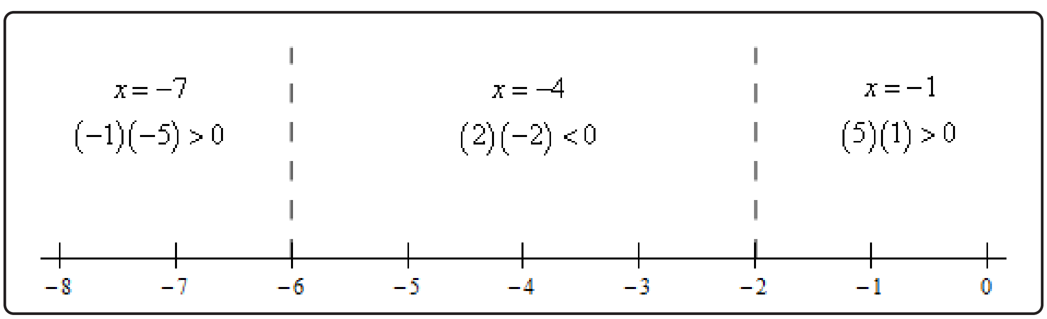
Hint

Knowing that the polynomial can only change sign at the points above how can we quickly determine if the polynomial is positive or negative in the ranges between those points?

Step 3

Recall from the discussion in the notes for this section that because the points from the previous step are the only places where the polynomial might change sign we can quickly determine if the polynomial is positive/negative in the ranges between each of these points simply by plugging in "test points" from each region into the polynomial to check the sign.

So, let's sketch a quick number line with the points where the polynomial is zero graphed on it. We'll also show the test point computations on the number line as well. Here is the number line.



Step 4

All we need to do now is get the solution from the number line in the previous step. Here is both the inequality and interval notation from of the answer.

$$\boxed{\begin{gathered}-6 < x < -2\\(-6,-2)\end{gathered}}$$

3. Solve the following inequality.

$$4t^2 \le 15 - 17t$$

Step 1

The first thing we need to do is get a zero on one side of the inequality and then, if possible, factor the polynomial.

$$\begin{gathered}4t^2 + 17t - 15 \le 0\\(t+5)(4t-3) \le 0\end{gathered}$$

Hint

Where are the only places where the polynomial might change signs?

Step 2

Despite the fact that this is an inequality we first need to know where the polynomial is zero. From the factored from we can quickly see that the polynomial will be zero at,

$$t = -5 \qquad t = \frac{3}{4}$$

Remember that these points are important because they are the only places where the polynomial on the left side of the inequality might change sign. Given that we want to know where the polynomial is zero (which we now know) or negative knowing where the polynomial might change sign will help considerably with determining the answer we're looking for.

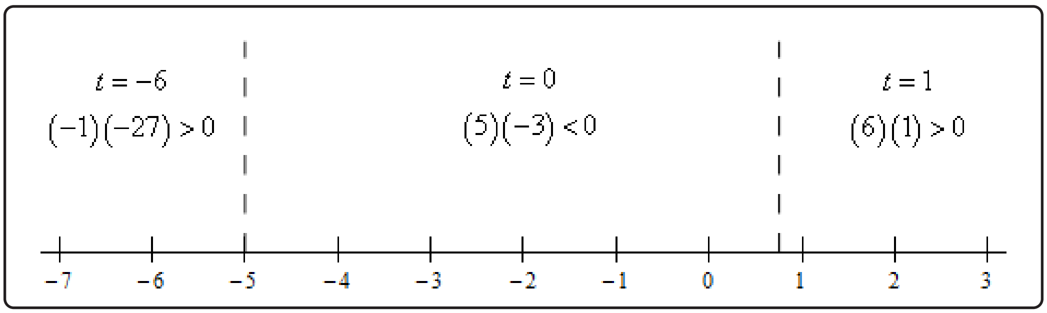
Hint

Knowing that the polynomial can only change sign at the points above how can we quickly determine if the polynomial is positive or negative in the ranges between those points?

Step 3

Recall from the discussion in the notes for this section that because the points from the previous step are the only places where the polynomial might change sign we can quickly determine if the polynomial is positive/negative in the ranges between each of these points simply by plugging in "test points" from each region into the polynomial to check the sign.

So, let's sketch a quick number line with the points where the polynomial is zero graphed on it. We'll also show the test point computations on the number line as well. Here is the number line.



Step 4

All we need to do now is get the solution from the number line in the previous step. Here is both the inequality and interval notation from of the answer.

$$\boxed{\begin{array}{c} -5 \le t \le \dfrac{3}{4} \\ \left[-5, \dfrac{3}{4}\right] \end{array}}$$

4. Solve the following inequality.

$$z^2 + 34 > 12z$$

Step 1

The first thing we need to do is get a zero on one side of the inequality and then, if possible, factor the polynomial.

$$z^2 - 12z + 34 > 0$$

In this case the polynomial doesn't factor.

Hint

Where are the only places where the polynomial might change signs?

Step 2

Despite the fact that this is an inequality we first need to know where the polynomial is zero. Because the polynomial didn't factor we'll need to use the quadratic formula to determine where it's zero.

$$z = \frac{12 \pm \sqrt{144 - 4(1)(34)}}{2(1)} = \frac{12 \pm \sqrt{8}}{2} \qquad \Rightarrow \qquad z = 4.5858,\ 7.4142$$

We'll need these points in decimal form to make the rest of the problem easier.

Remember that these points are important because they are the only places where the polynomial on the left side of the inequality might change sign. Given that we want to know where the polynomial is positive knowing where the polynomial might change sign will help considerably with determining the answer we're looking for.

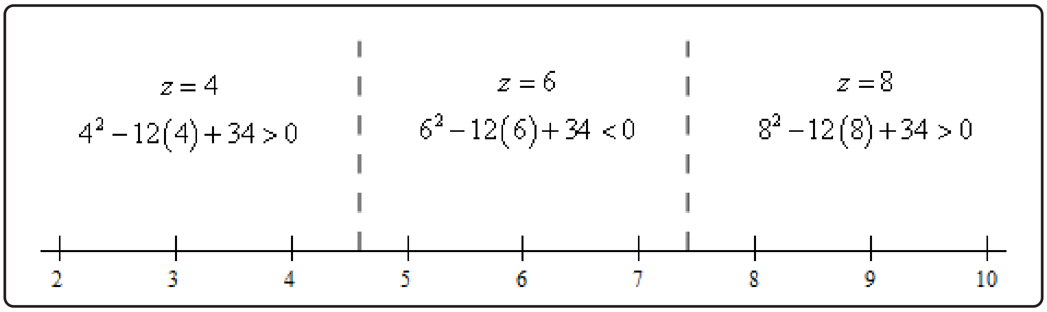
Hint

Knowing that the polynomial can only change sign at the points above how can we quickly determine if the polynomial is positive or negative in the ranges between those points?

Step 3

Recall from the discussion in the notes for this section that because the points from the previous step are the only places where the polynomial might change sign we can quickly determine if the polynomial is positive/negative in the ranges between each of these points simply by plugging in "test points" from each region into the polynomial to check the sign.

So, let's sketch a quick number line with the points where the polynomial is zero graphed on it. We'll also show the test point computations on the number line as well. Here is the number line.



Step 4

All we need to do now is get the solution from the number line in the previous step. Here is both the inequality and interval notation from of the answer.

$$z < 4.5858 \quad \text{and} \quad z > 7.4142$$
$$(-\infty, 4.5858) \quad \text{and} \quad (7.4142, \infty)$$

5. Solve the following inequality.

$$y^2 - 2y + 1 \le 0$$

Step 1

The first thing we need to do is get a zero on one side of the inequality (which is already done for this problem) and then, if possible, factor the polynomial.

$$(y-1)^2 \le 0$$

Hint

Where are the only places where the polynomial might change signs?

Step 2

Despite the fact that this is an inequality we first need to know where the polynomial is zero. From the factored from we can quickly see that the polynomial will be zero at,

$$y = 1$$

Hint

Is it possible for the polynomial to ever be negative?

Step 3

This problem works a little differently than the others in this section. Because the polynomial is a perfect square we know that it can never be negative! It is only possible for it to be zero or positive.

We are being asked to determine where the polynomial is negative or zero. As noted however it isn't possible for it to be negative. Therefore the only solution we can get for this inequality is where it is zero and we found that in the previous step.

The answer is then,

$$\boxed{y = 1}$$

In this case the answer is a single number and not an inequality. This happens on occasion and we shouldn't worry about these kinds of "unusual" answers.

6. Solve the following inequality.

$$t^4 + t^3 - 12t^2 < 0$$

Step 1

The first thing we need to do is get a zero on one side of the inequality (which is already done for this problem) and then, if possible, factor the polynomial.

$$\begin{aligned} t^2\left(t^2 + t - 12\right) &< 0 \\ t^2\left(t + 4\right)\left(t - 3\right) &< 0 \end{aligned}$$

Hint

Where are the only places where the polynomial might change signs?

Step 2

Despite the fact that this is an inequality we first need to know where the polynomial is zero. From the factored from we can quickly see that the polynomial will be zero at,

$$t = -4 \qquad t = 0 \qquad t = 3$$

Remember that these points are important because they are the only places where the polynomial on the left side of the inequality might change sign. Given that we want to know where the polynomial is negative knowing where the polynomial might change sign will help considerably with determining the answer we're looking for.

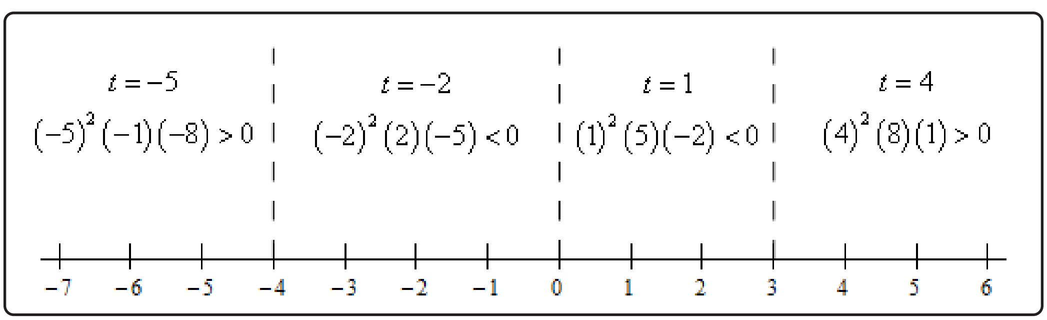
Hint

Knowing that the polynomial can only change sign at the points above how can we quickly determine if the polynomial is positive or negative in the ranges between those points?

Step 3

Recall from the discussion in the notes for this section that because the points from the previous step are the only places where the polynomial might change sign we can quickly determine if the polynomial is positive/negative in the ranges between each of these points simply by plugging in "test points" from each region into the polynomial to check the sign.

So, let's sketch a quick number line with the points where the polynomial is zero graphed on it. We'll also show the test point computations on the number line as well. Here is the number line.



Be careful with the first term in the factored form when plugging in the test points! It is squared and so will always be positive regardless of the sign of the test points. One of the bigger mistakes that students make with this kind of problem is to miss the square and treat that term as negative when plugging in a negative test point.

Step 4

All we need to do now is get the solution from the number line in the previous step. Here is both the inequality and interval notation from of the answer.

$$\boxed{\begin{array}{ccc} -4 < t < 0 & \text{and} & 0 < t < 3 \\ (-4,0) & \text{and} & (0,3) \end{array}}$$

Be careful with your answer here and don't include $t = 0$! It might be tempting to do that to "simplify" the answer into a single inequality/interval but the polynomial is zero at $t = 0$ and we only want to know where the polynomial is negative! Therefore we cannot include $t = 0$ in our answer and we'll need to write it as two inequalities/intervals.

2.13 Rational Inequalities

1. Solve the following inequality.

$$\frac{4-x}{x+3} > 0$$

Step 1

The first thing we need to do is get a zero on one side of the inequality and then, if possible, factor the numerator and denominator as much as possible.

For this problem we already have zero on one side of the inequality and there is no factoring to do with the problem.

Hint

Where are the only places where the rational expression might change signs?

Step 2

Recall from the discussion in the notes for this section that the rational expression can only change sign where the numerator is zero and/or where the denominator is zero.

We can see that the numerator will be zero at,

$$x = 4$$

and the denominator will be zero at,

$$x = -3$$

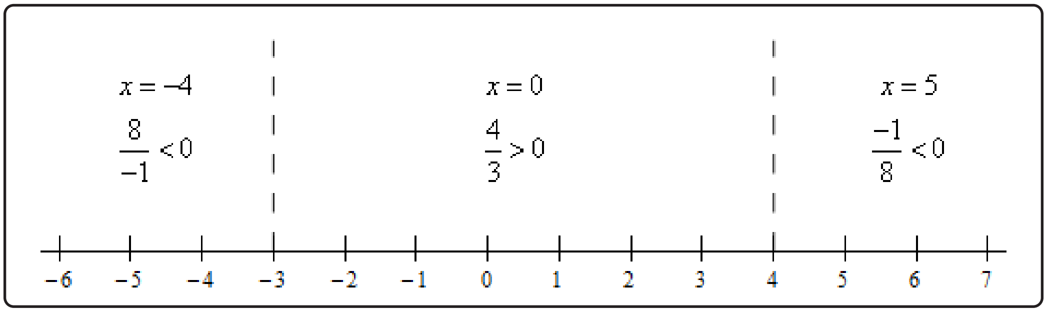
Hint

Knowing that the rational expression can only change sign at the points above how can we quickly determine if the rational expression is positive or negative in the ranges between those points?

Step 3

Just as we did with polynomial inequalities all we need to do is check the rational expression at test points in each region between the points from the previous step. The rational expression will have the same sign as the sign at the test point since it can only change sign at those points.

Here is a sketch of a number line with the points from the previous step graphed on it. We'll also show the test point computations on the number line as well. Here is the number line.



Step 4

All we need to do now is get the solution from the number line in the previous step. Here is both the inequality and interval notation from of the answer.

$$\boxed{\begin{array}{c} -3 < x < 4 \\ (-3, 4) \end{array}}$$

2. Solve the following inequality.

$$\frac{2z-5}{z-7} \le 0$$

Step 1

The first thing we need to do is get a zero on one side of the inequality and then, if possible, factor the numerator and denominator as much as possible.

For this problem we already have zero on one side of the inequality and there is no factoring to do with the problem.

Hint

Where are the only places where the rational expression might change signs?

Step 2

Recall from the discussion in the notes for this section that the rational expression can only change sign where the numerator is zero and/or where the denominator is zero.

We can see that the numerator will be zero at,

$$z = \frac{5}{2}$$

and the denominator will be zero at,

$$z = 7$$

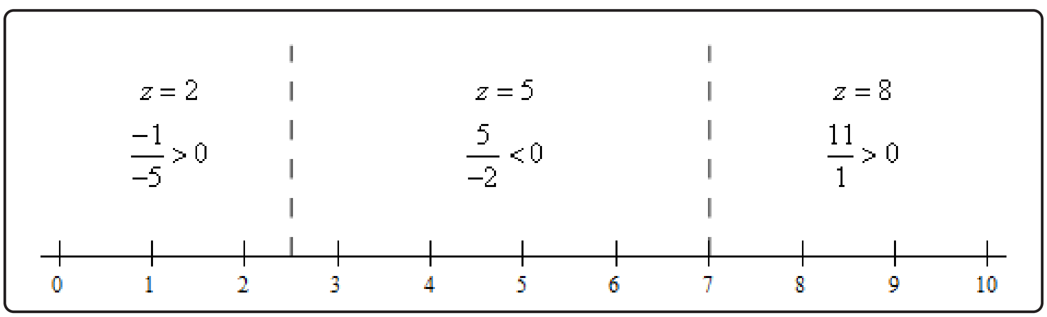
Hint

Knowing that the rational expression can only change sign at the points above how can we quickly determine if the rational expression is positive or negative in the ranges between those points?

Step 3

Just as we did with polynomial inequalities all we need to do is check the rational expression at test points in each region between the points from the previous step. The rational expression will have the same sign as the sign at the test point since it can only change sign at those points.

Here is a sketch of a number line with the points from the previous step graphed on it. We'll also show the test point computations on the number line as well. Here is the number line.



Step 4

All we need to do now is get the solution from the number line in the previous step. Here is both the inequality and interval notation from of the answer.

$$\boxed{\begin{array}{c}\frac{5}{2} \le z < 7 \\ \left[\frac{5}{2}, 7\right)\end{array}}$$

Be careful with the endpoints for this problem. Because we have an equal sign in the original inequality we need to include $z = \frac{5}{2}$ because the numerator and hence the rational expression will be zero there. However, we can't include $z = 7$ because the denominator is zero there and so the rational expression has division by zero at that point!

3. Solve the following inequality.

$$\frac{w^2 + 5w - 6}{w - 3} \ge 0$$

Step 1

The first thing we need to do is get a zero on one side of the inequality and then, if possible, factor the numerator and denominator as much as possible.

For this problem we already have zero on one side of the inequality but we do need to factor the numerator.

$$\frac{(w + 6)(w - 1)}{w - 3} \ge 0$$

Hint

Where are the only places where the rational expression might change signs?

Step 2

Recall from the discussion in the notes for this section that the rational expression can only change sign where the numerator is zero and/or where the denominator is zero.

We can see that the numerator will be zero at,

$$w = -6 \qquad\qquad w = 1$$

and the denominator will be zero at,

$$w = 3$$

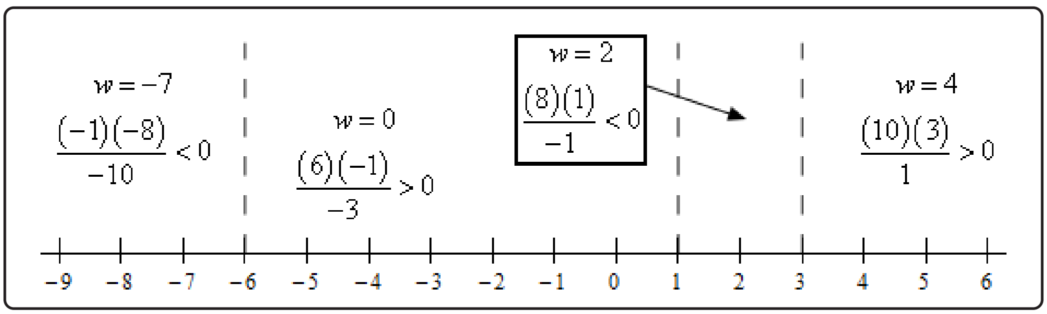
Hint

Knowing that the rational expression can only change sign at the points above how can we quickly determine if the rational expression is positive or negative in the ranges between those points?

Step 3

Just as we did with polynomial inequalities all we need to do is check the rational expression at test points in each region between the points from the previous step. The rational expression will have the same sign as the sign at the test point since it can only change sign at those points.

Here is a sketch of a number line with the points from the previous step graphed on it. We'll also show the test point computations on the number line as well. Here is the number line.



Step 4

All we need to do now is get the solution from the number line in the previous step. Here is both the inequality and interval notation from of the answer.

$$\begin{array}{ccc} -6 \le w \le 1 & \text{and} & w > 3 \\ [-6,1] & \text{and} & (3,\infty) \end{array}$$

Be careful with the endpoints for this problem. Because we have an equal sign in the original inequality we need to include $w = -6$ and $w = 1$ because the numerator and hence the rational expression will be zero there. However, we can't include $w = 3$ because the denominator is zero there and so the rational expression has division by zero at that point!

4. Solve the following inequality.

$$\frac{3x+8}{x-1} < -2$$

Step 1

The first thing we need to do is get a zero on one side of the inequality and then, if possible, factor the numerator and denominator as much as possible.

So, we first need to add 2 to both sides to get,

$$\frac{3x+8}{x-1} + 2 < 0$$

We now need to combine the two terms in to a single rational expression.

$$\begin{aligned}\frac{3x+8}{x-1} + \frac{2(x-1)}{x-1} &< 0\\ \frac{3x+8+2x-2}{x-1} &< 0\\ \frac{5x+6}{x-1} &< 0\end{aligned}$$

At this point we can also see that factoring will not be needed for this problem.

Hint

Where are the only places where the rational expression might change signs?

Step 2

Recall from the discussion in the notes for this section that the rational expression can only change sign where the numerator is zero and/or where the denominator is zero.

We can see that the numerator will be zero at,

$$x = -\frac{6}{5}$$

and the denominator will be zero at,

$$x = 1$$

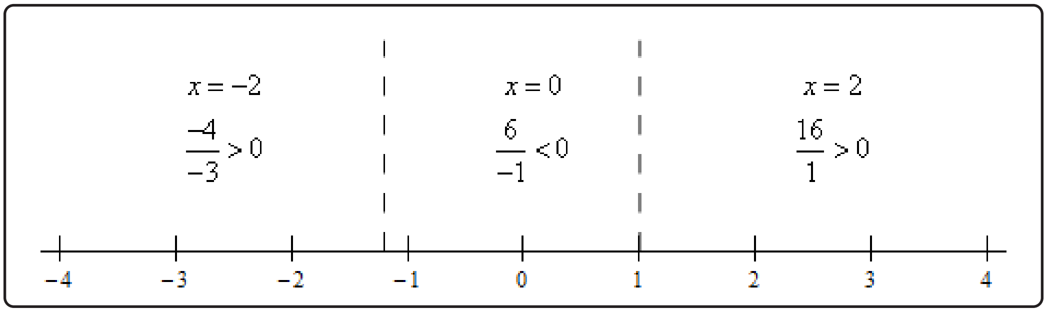
Hint

Knowing that the rational expression can only change sign at the points above how can we quickly determine if the rational expression is positive or negative in the ranges between those points?

Step 3

Just as we did with polynomial inequalities all we need to do is check the rational expression at test points in each region between the points from the previous step. The rational expression will have the same sign as the sign at the test point since it can only change sign at those points.

Here is a sketch of a number line with the points from the previous step graphed on it. We'll also show the test point computations on the number line as well. Here is the number line.



Step 4

All we need to do now is get the solution from the number line in the previous step. Here is both the inequality and interval notation from of the answer.

$$\boxed{\begin{array}{c}-\frac{6}{5}<x<1\\ \left(-\frac{6}{5},1\right)\end{array}}$$

5. Solve the following inequality.

$$u \le \frac{4}{u-3}$$

Step 1

The first thing we need to do is get a zero on one side of the inequality and then, if possible, factor the numerator and denominator as much as possible.

So, we first need to get zero on one side of the inequality.

$$u - \frac{4}{u-3} \le 0$$

We now need to combine the two terms in to a single rational expression.

$$\begin{aligned} \frac{u\left(u-3\right)}{u-3} - \frac{4}{u-3} &\le 0 \\ \frac{u^2 - 3u - 4}{u-3} &\le 0 \end{aligned}$$

Finally, we need to factor the numerator.

$$\frac{\left(u-4\right)\left(u+1\right)}{u-3} \le 0$$

Hint

Where are the only places where the rational expression might change signs?

Step 2

Recall from the discussion in the notes for this section that the rational expression can only change sign where the numerator is zero and/or where the denominator is zero.

We can see that the numerator will be zero at,

$$u = -1 \qquad\qquad u = 4$$

and the denominator will be zero at,

$$u = 3$$

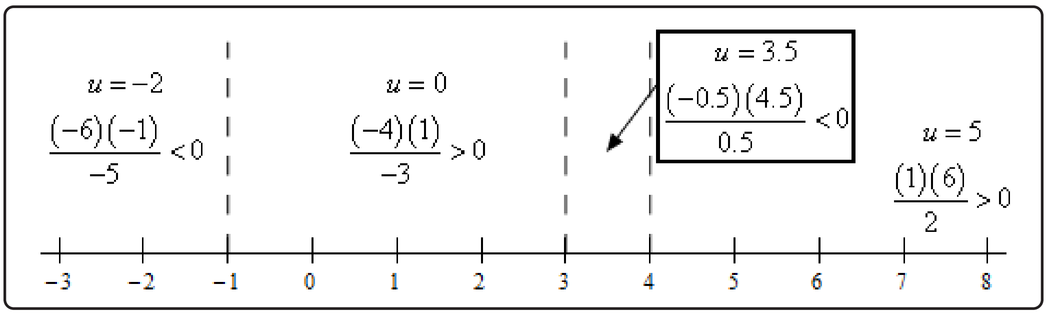
Hint

Knowing that the rational expression can only change sign at the points above how can we quickly determine if the rational expression is positive or negative in the ranges between those points?

Step 3

Just as we did with polynomial inequalities all we need to do is check the rational expression at test points in each region between the points from the previous step. The rational expression will have the same sign as the sign at the test point since it can only change sign at those points.

Here is a sketch of a number line with the points from the previous step graphed on it. We'll also show the test point computations on the number line as well. Here is the number line.



Step 4

All we need to do now is get the solution from the number line in the previous step. Here is both the inequality and interval notation from of the answer.

$$u \le -1 \quad \text{and} \quad 3 < u \le 4$$
$$(-\infty, -1] \quad \text{and} \quad (3, 4]$$

Be careful with the endpoints for this problem. Because we have an equal sign in the original inequality we need to include $u = -1$ and $u = 4$ because the numerator and hence the rational expression will be zero there. However, we can't include $u = 3$ because the denominator is zero there and so the rational expression has division by zero at that point!

6. Solve the following inequality.

$$\frac{t^3 - 6t^2}{t - 2} > 0$$

Step 1

The first thing we need to do is get a zero on one side of the inequality and then, if possible, factor the numerator and denominator as much as possible.

We already have zero on one side of the inequality but we still need to factor the numerator.

$$\frac{t^2\left(t-6\right)}{t-2}>0$$

Hint

Where are the only places where the rational expression might change signs?

Step 2

Recall from the discussion in the notes for this section that the rational expression can only change sign where the numerator is zero and/or where the denominator is zero.

We can see that the numerator will be zero at,

$$t=0 \qquad\qquad t=6$$

and the denominator will be zero at,

$$t=2$$

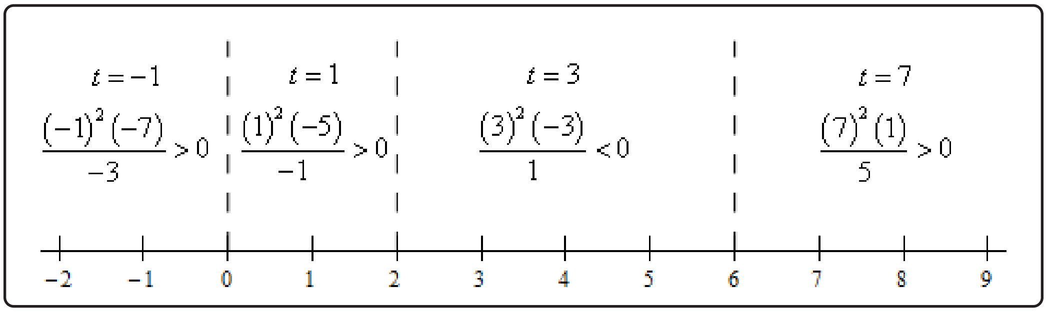
Hint

Knowing that the rational expression can only change sign at the points above how can we quickly determine if the rational expression is positive or negative in the ranges between those points?

Step 3

Just as we did with polynomial inequalities all we need to do is check the rational expression at test points in each region between the points from the previous step. The rational expression will have the same sign as the sign at the test point since it can only change sign at those points.

Here is a sketch of a number line with the points from the previous step graphed on it. We'll also show the test point computations on the number line as well. Here is the number line.


$t=-1$
$\frac{(-1)^2(-7)}{-3}>0$
$t=1$
$\frac{(1)^2(-5)}{-1}>0$
$t=3$
$\frac{(3)^2(-3)}{1}<0$
$t=7$
$\frac{(7)^2(1)}{5}>0$
−2
−1
0
1
2
3
4
5
6
7
8
9

Step 4

All we need to do now is get the solution from the number line in the previous step. Here is both the inequality and interval notation from of the answer.

$$\boxed{\begin{array}{cccc} t < 0, & 0 < t < 2 & \text{and} & t > 6 \\ (-\infty, 0)\,, & (0, 2) & \text{and} & (6, \infty) \end{array}}$$

Be careful to not include $t = 0$ in the answer! It might be tempting to "simplify" the first two inequalities in our answer into a single inequality. However, we're looking for where the rational expression is positive only and at $t = 0$ the rational expression is zero and so we need to exclude $t = 0$ from our answer.

2.14 Absolute Value Equations

1. Solve the following equation.

$$|4p - 7| = 3$$

Step 1

There really isn't all that much to this problem. All we need to do is use the formula we discussed in the notes for this section. Doing that gives,

$$4p - 7 = -3 \qquad \text{or} \qquad 4p - 7 = 3$$

Do not make the common mistake of just turning every minus sign inside the absolute value bars into a plus sign. That is just not how these work. The only way for the value of the absolute value to be 3 is for the quantity inside to be either -3 or 3. In other words, we get rid of the absolute value bars by using the formula from the notes.

Step 2

At this point all we need to do is solve each of the linear equations we got in the previous step. Doing that gives,

$$\begin{aligned} 4p &= 4 & \quad\text{or}\quad && 4p &= 10 \\ p &= 1 & \quad\text{or}\quad && p &= \frac{10}{4} = \frac{5}{2} \end{aligned}$$

The two solutions are then : $\boxed{p = 1 \text{ and } p = \frac{5}{2}}$.

---

2. Solve the following equation.

$$|2 - 4x| = 1$$

Step 1

There really isn't all that much to this problem. All we need to do is use the formula we discussed in the notes for this section. Doing that gives,

$$2 - 4x = -1 \qquad \text{or} \qquad 2 - 4x = 1$$

Do not make the common mistake of just turning every minus sign inside the absolute value bars into a plus sign. That is just not how these work. The only way for the value of the absolute value to be 1 is for the quantity inside to be either -1 or 1. In other words, we get rid of the absolute value bars by using the formula from the notes.

Step 2

At this point all we need to do is solve each of the linear equations we got in the previous step. Doing that gives,

$$\begin{aligned} -4x &= -3 \quad &\text{or}\quad -4x &= -1 \\ x &= \frac{3}{4} \quad &\text{or}\quad x &= \frac{1}{4} \end{aligned}$$

The two solutions are then : $\boxed{x = \frac{1}{4} \text{ and } x = \frac{3}{4}}$.

3. Solve the following equation.

$$6u = |1 + 3u|$$

Hint

Just because the quantity outside of the absolute value bars in not a number does not mean this problem works any differently. Just remember to be careful with your answers!

Step 1

Despite the fact that the quantity outside of the absolute value bars is not a positive number doesn't mean that we can't use the same process that we used in the first two problems.

Using the formula from the notes gives,

$$1 + 3u = -6u \quad \text{or} \quad 1 + 3u = 6u$$

Step 2

Now solving each these linear equations gives,

$$\begin{aligned} 1 + 3u &= -6u \quad &\text{or}\quad 1 + 3u &= 6u \\ 1 &= -9u \quad &\text{or}\quad 1 &= 3u \\ u &= -\frac{1}{9} \quad &\text{or}\quad u &= \frac{1}{3} \end{aligned}$$

Step 3

Now, because the quantity outside of the absolute value bars was not a positive constant we need to be careful with the answers we got in the previous step. It is possible that one or both are not in fact solutions to the original equation. So, we need to verify each of the possible solutions from the previous step by checking them in the original equation.

$$u = -\frac{1}{9}: \quad 6\left(-\frac{1}{9}\right) \stackrel{?}{=} \left|1 + 3\left(-\frac{1}{9}\right)\right| \quad \to \quad -\frac{2}{3} \stackrel{?}{=} \left|\frac{2}{3}\right| \quad \to \quad -\frac{2}{3} \neq \frac{2}{3} \quad \text{NOT OK}$$

$$u = \frac{1}{3}: \quad 6\left(\frac{1}{3}\right) \stackrel{?}{=} \left|1 + 3\left(\frac{1}{3}\right)\right| \quad \to \quad 2 \stackrel{?}{=} |2| \quad \to \quad 2 = 2 \quad \text{OK}$$

Therefore, the only solution to the original equation is then : $\boxed{u = \frac{1}{3}}$.

---

4. Solve the following equation.

$$|2x - 3| = 4 - x$$

Hint

Just because the quantity outside of the absolute value bars in not a number does not mean this problem works any differently. Just remember to be careful with your answers!

Step 1

Despite the fact that the quantity outside of the absolute value bars is not a positive number doesn't mean that we can't use the same process that we used in the first two problems.

Using the formula from the notes gives,

$$2x - 3 = -(4 - x) = x - 4 \quad \text{or} \quad 2x - 3 = 4 - x$$

Step 2

Now solving each these linear equations gives,

$$\begin{aligned} 2x - 3 &= x - 4 & \text{or} \quad 2x - 3 &= 4 - x \\ x &= -1 & \text{or} \quad 3x &= 7 \\ x &= -1 & \text{or} \quad x &= \frac{7}{3} \end{aligned}$$

Step 3

Now, because the quantity outside of the absolute value bars was not a positive constant we need to be careful with the answers we got in the previous step. It is possible that one or both are not in fact solutions to the original equation. So, we need to verify each of the possible solutions from the previous step by checking them in the original equation.

$$x=-1: \quad \left|2\left(-1\right)-3\right| \stackrel{?}{=} 4-\left(-1\right) \quad \rightarrow \quad \left|-5\right| \stackrel{?}{=} 5 \quad \rightarrow \quad 5=5 \quad \text{OK}$$

$$x=\frac{7}{3}: \quad \left|2\left(\frac{7}{3}\right)-3\right| \stackrel{?}{=} 4-\left(\frac{7}{3}\right) \quad \rightarrow \quad \left|\frac{5}{3}\right| \stackrel{?}{=} \frac{5}{3} \quad \rightarrow \quad \frac{5}{3}=\frac{5}{3} \quad \text{OK}$$

Therefore, the two solutions to the original equation are then : $\boxed{x=-1 \text{ and } x=\frac{7}{3}}$

.

5. Solve the following equation.

$$\left|\frac{1}{2}z+4\right| = \left|4z-6\right|$$

Hint

This problem works the same as all the others in this section do.

Step 1

This problem works identically to all the problems in this section. The only way the two absolute values can be equal is if the quantities inside them are the same value or the same value except for opposite signs. Doing this gives,

$$\frac{1}{2}z+4=-\left(4z-6\right)=6-4z \qquad \text{or} \qquad \frac{1}{2}z+4=4z-6$$

In other words, we can use the formula discussed in this section to do this problem!

Step 2

Now solving each these linear equations gives,

$$\begin{aligned} \frac{1}{2}z + 4 &= 6 - 4z & \text{or} && \frac{1}{2}z + 4 &= 4z - 6 \\ \frac{9}{2}z &= 2 & \text{or} && -\frac{7}{2}z &= -10 \\ z &= \frac{4}{9} & \text{or} && z &= \frac{20}{7} \end{aligned}$$

Now, because both sides of the equation have absolute values, we know that regardless of the value of x we plug into the original equation the absolute value will guarantee that the result will be positive and so we don't need to verify either of these solutions.

Therefore, the two solutions to the original equation are then : $\boxed{z = \frac{4}{9} \quad \text{and} \quad z = \frac{20}{7}}$

.

---

6. Find all the real valued solutions to the equation.

$$\left|x^2 + 2x\right| = 15$$

Hint

Don't let the fact that there is a quadratic term in the absolute value throw you off. This problem works exactly the same as the previous problems!

Step 1

To this point we've only worked problems that have linear terms in the absolute value bars. In this case we have a quadratic in the absolute value bars. That doesn't change how the problem works however. We work this exactly like the previous problems.

Applying the formula from this section gives,

$$x^2 + 2x = -15 \qquad \text{or} \qquad x^2 + 2x = 15$$

Step 2

To finish this problem all we need to do is solve each of the quadratic equations we got in the previous step.

Here is the solution to the first one given above.

$$x^2 + 2x = -15$$
$$x^2 + 2x + 15 = 0 \qquad \to \qquad x = \frac{-2 \pm \sqrt{4 - 4(1)(15)}}{2(1)} = \frac{-2 \pm \sqrt{56}\,i}{2}$$

Note that the instructions asked for "**real valued solutions**". This basically means that we don't want complex solutions and the solutions to the first quadratic are clearly complex and so we won't use them in our solution.

The solution to the second quadratic is,

$$x^2 + 2x - 15 = 0$$
$$(x+5)(x-3) = 0 \qquad \to \qquad x = -5, \quad x = 3$$

Both of these are real solutions and so are acceptable solutions.

Therefore, the two solutions to the original equation are then : $\boxed{x = -5 \quad \text{and} \quad x = 3}$
.

7. Find all the real valued solutions to the equation.

$$\left|x^2 + 4\right| = 1$$

Hint

Don't let the fact that there is a quadratic term in the absolute value throw you off. This problem works exactly the same as the previous problems!

Step 1

To this point we've only worked problems that have linear terms in the absolute value bars. In this case we have a quadratic in the absolute value bars. That doesn't change how the problem works however. We work this exactly like the previous problems.

Applying the formula from this section gives,

$$x^2 + 4 = -1 \qquad \text{or} \qquad x^2 + 4 = 1$$

Step 2

To finish this problem all we need to do is solve each of the quadratic equations we got in the previous step. Here is the solution to each of them.

$$x^2+4=-1 \quad\rightarrow\quad x^2=-5 \quad\rightarrow\quad x=\pm\sqrt{5}\,i$$
$$x^2+4=1 \quad\rightarrow\quad x^2=-3 \quad\rightarrow\quad x=\pm\sqrt{3}\,i$$

Note that the instructions asked for "**real valued solutions**". This basically means that we don't want complex solutions and the solutions to both of the quadratic equations from the first step are complex and so, for this equation, there are **no solutions**.

2.15 Absolute Value Inequalities

1. Solve the following equation.

$$|4t + 9| < 3$$

Step 1

There really isn't all that much to this problem. All we need to do is use the formula for "less than" inequalities we discussed in the notes for this section. Doing that gives,

$$-3 < 4t + 9 < 3$$

Step 2

To get the solution all we need to do then is solve the double inequality from the previous step. Here is that work.

$$-3 < 4t + 9 < 3$$
$$-12 < 4t < -6$$
$$\boxed{-3 < t < -\frac{3}{2}}$$

2. Solve the following equation.

$$|6 - 5x| \le 10$$

Step 1

There really isn't all that much to this problem. All we need to do is use the formula for "less than" inequalities we discussed in the notes for this section. Doing that gives,

$$-10 \le 6 - 5x \le 10$$

Step 2

To get the solution all we need to do then is solve the double inequality from the previous step. Here is that work.

$$-10 \le 6 - 5x \le 10$$
$$-16 \le -5x \le 4$$
$$\boxed{\frac{16}{5} \ge x \ge -\frac{4}{5}}$$

Remember that when dividing all parts of an inequality by a negative number (as we did here) we need to also switch the direction of the inequalities!

3. Solve the following equation.

$$|12x + 1| \le -9$$

Solution

There is **no solution** to this inequality.

We know that absolute value will only give positive or zero answers and so this inequality is asking what values of x will give a value on the left side (after taking the absolute value of course) that is less than a -9. In other words, any solution requires that the absolute value give a negative number and we know that can't happen. Therefore, there are no solutions to this inequality. This kinds of thing happens occasionally so don't get too excited about it when it does.

4. Solve the following equation.

$$|2w - 1| < 1$$

Step 1

There really isn't all that much to this problem. All we need to do is use the formula for "less than" inequalities we discussed in the notes for this section. Doing that gives,

$$-1 < 2w - 1 < 1$$

Step 2

To get the solution all we need to do then is solve the double inequality from the previous step. Here is that work.

$$-1 < 2w - 1 < 1$$
$$0 < 2w < 2$$
$$\boxed{0 < w < 1}$$

5. Solve the following equation.

$$|2z - 7| > 1$$

Step 1

There really isn't all that much to this problem. All we need to do is use the formula for "greater than" inequalities we discussed in the notes for this section. Doing that gives,

$$2z - 7 < -1 \quad \text{or} \quad 2z - 7 > 1$$

Step 2

To get the solution all we need to do then is solve the two inequalities from the previous step. Here is that work.

$$\begin{aligned} 2z - 7 < -1 \quad &\text{or} \quad 2z - 7 > 1 \\ 2z < 6 \quad &\text{or} \quad 2z > 8 \\ \boxed{z < 3 \quad \text{or} \quad z > 4} \end{aligned}$$

6. Solve the following equation.

$$|10 - 3w| \ge 4$$

Step 1

There really isn't all that much to this problem. All we need to do is use the formula for "greater than" inequalities we discussed in the notes for this section. Doing that gives,

$$10 - 3w \le -4 \quad \text{or} \quad 10 - 3w \ge 4$$

Step 2

To get the solution all we need to do then is solve the two inequalities from the previous step. Here is that work.

$$\begin{aligned} 10 - 3w \le -4 \quad &\text{or} \quad 10 - 3w \ge 4 \\ -3w \le -14 \quad &\text{or} \quad -3w \ge -6 \\ \boxed{w \ge \frac{14}{3} \quad \text{or} \quad w \le 2} \end{aligned}$$

Remember that when dividing all parts of an inequality by a negative number (as we did here) we need to also switch the direction of the inequalities!

7. Solve the following equation.

$$|4-3z|>7$$

Step 1

There really isn't all that much to this problem. All we need to do is use the formula for "greater than" inequalities we discussed in the notes for this section. Doing that gives,

$$4-3z<-7 \quad \text{or} \quad 4-3z>7$$

Step 2

To get the solution all we need to do then is solve the two inequalities from the previous step. Here is that work.

$$\begin{aligned} 4-3z &< -7 & \quad\text{or}\quad & 4-3z>7 \\ -3z &< -11 & \quad\text{or}\quad & -3z>3 \end{aligned}$$

$$\boxed{z>\frac{11}{3} \quad \text{or} \quad z<-1}$$

Remember that when dividing all parts of an inequality by a negative number (as we did here) we need to also switch the direction of the inequalities!

3 Graphing and Functions

In this chapter we will be introducing two topics that are very important in an algebra class. We will start off the chapter with a brief discussion of graphing including graphing lines and circles. This is not really the main topic of this chapter, but we need the basics down before moving into the second topic of this chapter. The next chapter will contain the remainder of the graphing discussion.

The second topic that we'll be looking at is that of functions. This is probably one of the more important ideas that will come out of an Algebra class. When first studying the concept of functions many students don't really understand the importance or usefulness of functions and function notation. The importance and/or usefulness of functions and function notation will only become apparent in later chapters and later classes. In fact, there are some topics that can only be done easily with function and function notation.

We'll discuss the formal definition of a function, how to graph functions and the various ways to combine functions. We'll also introduce the idea of an inverse function.

The following sections are the practice problems, with solutions, for this material.

3.1 Graphing

1. Construct a table of at least 4 ordered pairs of points on the graph of the following equation and use the ordered pairs from the table to sketch the graph of the equation.

$$y = 3x + 4$$

Hint

If you don't know what the graph of a given equation is it can be very difficult to determine a good selection of values of x to use to construct the table. For this equation try a selection of at least a couple of points to either side of zero (maybe even including zero).

Step 1

It is always a little difficult to know just what a good selection of values of x to use to determine the ordered pairs we will use to sketch the graph of an equation if you don't know just what the graph looks like. Eventually you'll do enough problems that you'll start to develop some intuition on just what good values to try are for many equations.

For this equation a selection of points on either side of zero should be sufficient to get an idea of what the graph of this equation looks like. We'll also include zero for no other reason that it will give an extra point on the graph.

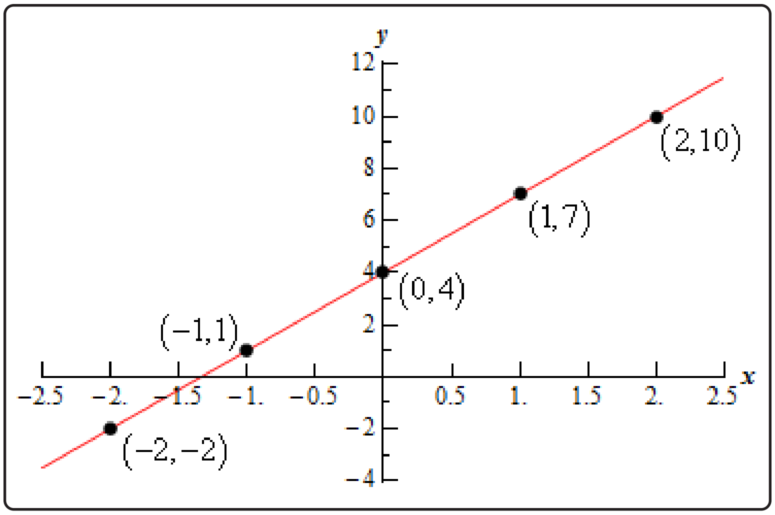
Here is the table of points we'll use for this problem.

x	y	(x, y)
-2	-2	$(-2, -2)$
-1	1	$(-1, 1)$
0	4	$(0, 4)$
1	7	$(1, 7)$
2	10	$(2, 10)$

We'll leave the actual computations to you to verify but recall that all we do is take the x and plug it into the equation to determine the corresponding y value and then form the ordered pair for the x and its corresponding y value.

Step 2

Here is a sketch of the equation.



2. Construct a table of at least 4 ordered pairs of points on the graph of the following equation and use the ordered pairs from the table to sketch the graph of the equation.

$$y = 1 - x^2$$

Hint

If you don't know what the graph of a given equation is it can be very difficult to determine a good selection of values of x to use to construct the table. For this equation try a selection of at least a couple of points to either side of zero (maybe even including zero).

Step 1

It is always a little difficult to know just what a good selection of values of x to use to determine the ordered pairs we will use to sketch the graph of an equation if you don't know just what the graph looks like. Eventually you'll do enough problems that you'll start to develop some intuition on just what good values to try are for many equations.

For this equation a selection of points on either side of zero should be sufficient to get an idea of what the graph of this equation looks like. We'll also include zero for no other reason that it will give an extra point on the graph.

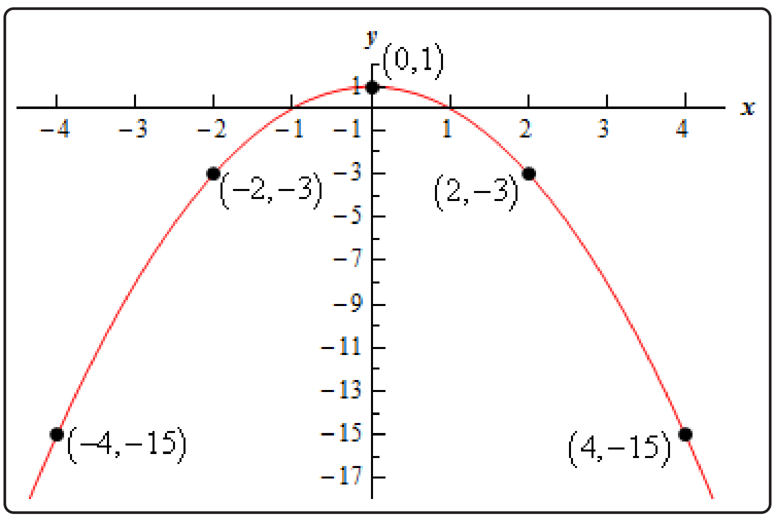
Here is the table of points we'll use for this problem.

x	y	(x, y)
-4	-15	$(-4, -15)$
-2	-3	$(-2, -3)$
0	1	$(0, 1)$
2	-3	$(2, -3)$
4	-15	$(4, -15)$

We'll leave the actual computations to you to verify but recall that all we do is take the x and plug it into the equation to determine the corresponding y value and then form the ordered pair for the x and its corresponding y value.

Step 2

Here is a sketch of the equation.



3. Construct a table of at least 4 ordered pairs of points on the graph of the following equation and use the ordered pairs from the table to sketch the graph of the equation.

$$y = 2 + \sqrt{x}$$

Hint

If you don't know what the graph of a given equation is it can be very difficult to determine a good selection of values of x to use to construct the table. For this equation remember where the square root is defined and remember that we want to evaluate the equation so pick values of x that will make this easy.

Step 1

It is always a little difficult to know just what a good selection of values of x to use to determine the ordered pairs we will use to sketch the graph of an equation if you don't know just what the graph looks like. Eventually you'll do enough problems that you'll start to develop some intuition on just what good values to try are for many equations.

For this equation we need to recall that we can only plug positive values of x or zero into the square root so we'll need to avoid negatives values of x. Also, because we want to actually be able to quickly evaluate the square root and get values of y that will be easy to graph we'll stick with perfect squares for our choices of x.

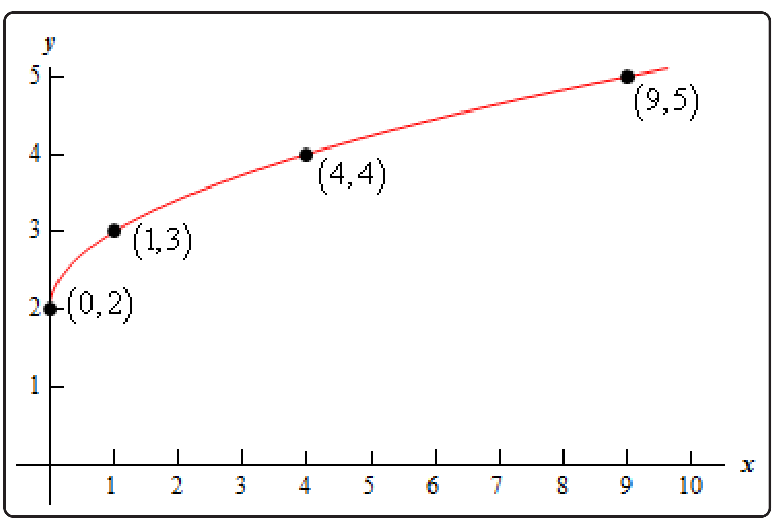
Here is the table of points we'll use for this problem.

x	y	(x,y)
0	2	$(0,2)$
1	3	$(1,3)$
4	4	$(4,4)$
9	5	$(9,5)$

We'll leave the actual computations to you to verify but recall that all we do is take the x and plug it into the equation to determine the corresponding y value and then form the ordered pair for the x and its corresponding y value.

Step 2

Here is a sketch of the equation.



4. Determine the x-intercepts and the y-intercepts for the following equation.

$$3x - 7y = 10$$

Step 1

Recall that in order to find the y-intercept all we need to do is plug $x = 0$ into the equation and solve for y. Doing that for this equation gives,

$$\begin{aligned} 3\left(0\right) - 7y &= 10 \\ -7y &= 10 \\ y &= -\frac{10}{7} \end{aligned}$$

The y-intercept for this equation is then the point : $\left(0, -\frac{10}{7}\right)$.

Step 2

Finding the x-intercept is similar to the y-intercept. All we do is plug in $y = 0$ and solve for x. Doing that for this equation gives,

$$\begin{aligned} 3x - 7\left(0\right) &= 10 \\ 3x &= 10 \\ x &= \frac{10}{3} \end{aligned}$$

The x-intercept for this equation is then the point : $\left(\frac{10}{3}, 0\right)$.

5. Determine the x-intercepts and the y-intercepts for the following equation.

$$y = 6 - x^2$$

Step 1

Recall that in order to find the y-intercept all we need to do is plug $x = 0$ into the equation and solve for y. Doing that for this equation gives,

$$\begin{aligned} y &= 6 - \left(0\right)^2 \\ y &= 6 \end{aligned}$$

The y-intercept for this equation is then the point : $\left(0, 6\right)$.

Step 2

Finding the x-intercept is similar to the y-intercept. All we do is plug in $y = 0$ and solve for x. Doing that for this equation gives,

$$\begin{aligned} 0 &= 6 - x^2 \\ x^2 &= 6 \\ x &= \pm\sqrt{6} \end{aligned}$$

The x-intercepts for this equation are then the two points : $\left(-\sqrt{6}, 0\right)$ and $\left(\sqrt{6}, 0\right)$.

---

6. Determine the x-intercepts and the y-intercepts for the following equation.

$$y = x^2 + 6x - 7$$

Step 1

Recall that in order to find the y-intercept all we need to do is plug $x = 0$ into the equation and solve for y. Doing that for this equation gives,

$$\begin{aligned} y &= (0)^2 + 6(0) - 7 \\ y &= -7 \end{aligned}$$

The y-intercept for this equation is then the point : $(0, -7)$.

Step 2

Finding the x-intercept is similar to the y-intercept. All we do is plug in $y = 0$ and solve for x. Doing that for this equation gives,

$$\begin{aligned} 0 &= x^2 + 6x - 7 \\ 0 &= (x + 7)(x - 1) \qquad \Rightarrow \qquad x = -7, \quad x = 1 \end{aligned}$$

The x-intercepts for this equation are then the two points : $(-7, 0)$ and $(1, 0)$.

---

7. Determine the x-intercepts and the y-intercepts for the following equation.

$$y = x^2 + 10$$

Step 1

Recall that in order to find the y-intercept all we need to do is plug $x = 0$ into the equation and solve for y. Doing that for this equation gives,

$$\begin{aligned} y &= (0)^2 + 10 \\ y &= 10 \end{aligned}$$

The y-intercept for this equation is then the point : $(0, 10)$.

Step 2

Finding the x-intercept is similar to the y-intercept. All we do is plug in $y = 0$ and solve for x. Doing that for this equation gives,

$$\begin{aligned} 0 &= x^2 + 10 \\ x^2 &= -10 \\ x &= \pm\sqrt{-10} = \pm\sqrt{10}\,i \end{aligned}$$

Because we got complex solutions to this equation we know that this equation has **no x-intercepts**.

8. Determine the x-intercepts and the y-intercepts for the following equation.

$$y = x^2 + 6x + 58$$

Step 1

Recall that in order to find the y-intercept all we need to do is plug $x = 0$ into the equation and solve for y. Doing that for this equation gives,

$$\begin{aligned} y &= (0)^2 + 6(0) + 58 \\ y &= 58 \end{aligned}$$

The y-intercept for this equation is then the point : $(0, 58)$.

Step 2

Finding the x-intercept is similar to the y-intercept. All we do is plug in $y = 0$ and solve for

x. Doing that for this equation gives,

$$\begin{aligned}0 &= x^2 + 6x + 58 \\ x &= \frac{-6 \pm \sqrt{6^2 - 4(1)(58)}}{2(1)} = \frac{-6 \pm \sqrt{-196}}{2} = \frac{-6 \pm 14\,i}{2} = -3 \pm 7i\end{aligned}$$

Because we got complex solutions to this equation we know that this equation has **no x-intercepts**.

9. Determine the x-intercepts and the y-intercepts for the following equation.

$$y = (x+3)^2 - 8$$

Step 1

Recall that in order to find the y-intercept all we need to do is plug $x = 0$ into the equation and solve for y. Doing that for this equation gives,

$$\begin{aligned}y &= (0+3)^2 - 8 \\ y &= 1\end{aligned}$$

The y-intercept for this equation is then the point : $(0, 1)$.

Step 2

Finding the x-intercept is similar to the y-intercept. All we do is plug in $y = 0$ and solve for x. Doing that for this equation gives,

$$\begin{aligned}0 &= (x+3)^2 - 8 \\ (x+3)^2 &= 8 \\ x + 3 &= \pm\sqrt{8} \\ x &= -3 \pm \sqrt{8}\end{aligned}$$

The x-intercepts for this equation are then the two points : $\left(-3 - \sqrt{8}, 0\right)$ and $\left(-3 + \sqrt{8}, 0\right)$.

Don't worry about the "messy" answers here. This kind of intercept will show up occasionally so we need to be able to deal with them when they do.

3.2 Lines

1. Determine the slope of the line containing the two points below and sketch the graph of the line.

$$(-2,4)\,,\quad (1,10)$$

Step 1

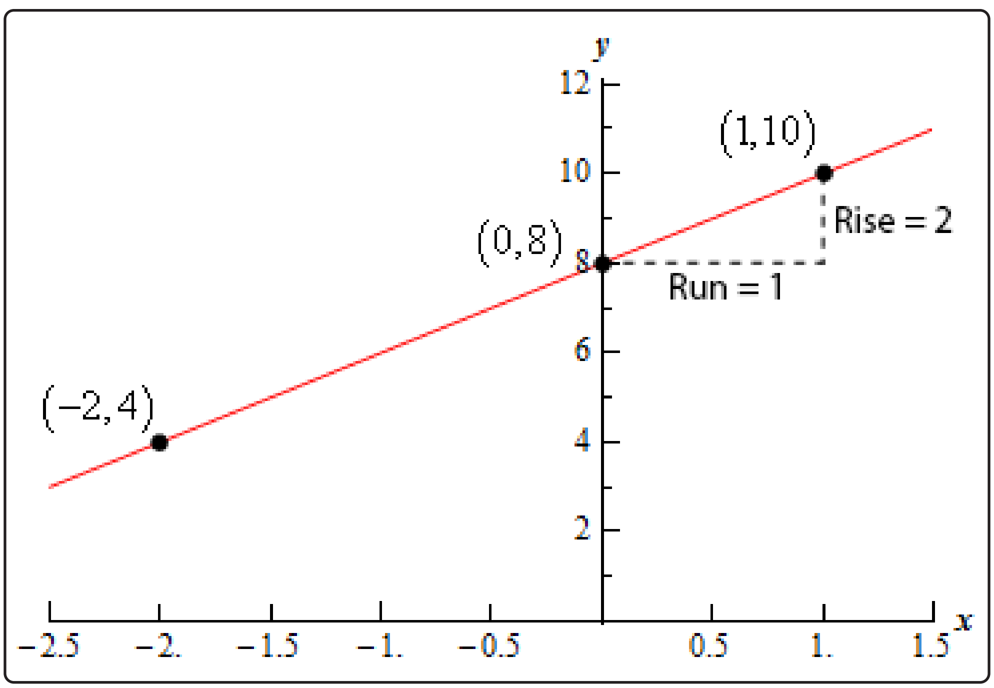
Let's find the slope of the line. We'll let the first point listed above be the point (x_1,y_1) and the second point listed be the point (x_2,y_2) in the slope formula. Note that it doesn't really matter which point is which. All that matters is that you stay consistent when you plug values into the formula.

Here's the slope.

$$m = \frac{10-4}{1-(-2)} = \frac{6}{3} = 2$$

Step 2

Here is a sketch of the line.



We've included an extra point, $(0,8)$, to help illustrate the slope we computed in Step 1.

2. Determine the slope of the line containing the two points below and sketch the graph of the line.

$$(8,2)\,,\quad (14,-7)$$

Step 1

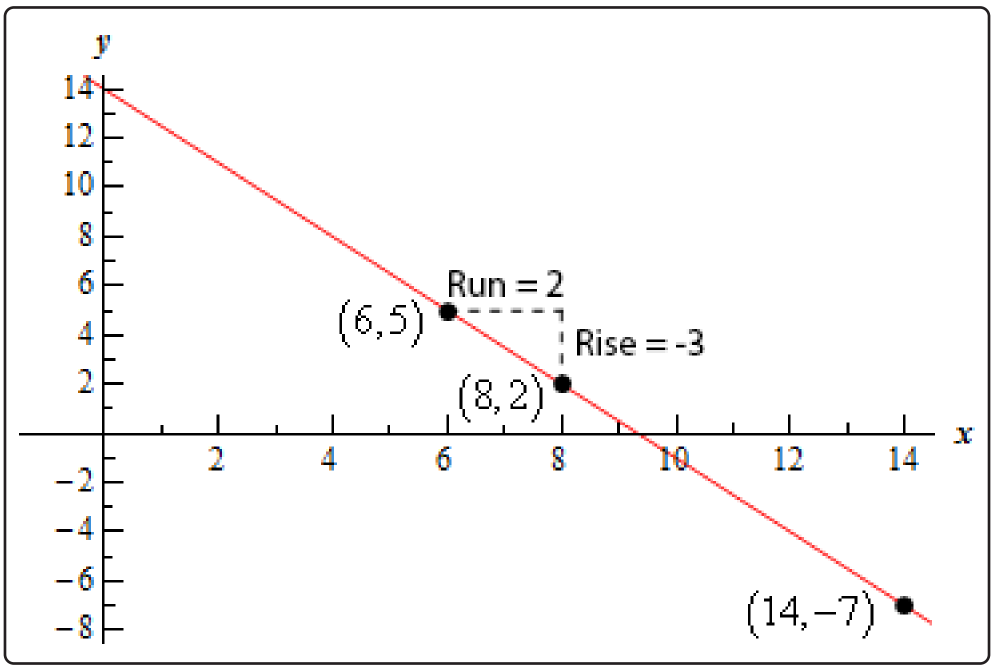
Let's find the slope of the line. We'll let the first point listed above be the point (x_1, y_1) and the second point listed be the point (x_2, y_2) in the slope formula. Note that it doesn't really matter which point is which. All that matters is that you stay consistent when you plug values into the formula.

Here's the slope.

$$m = \frac{-7-2}{14-8} = \frac{-9}{6} = -\frac{3}{2}$$

Step 2

Here is a sketch of the line.



We've included an extra point, $(6,5)$, to help illustrate the slope we computed in Step 1.

3. Write down the equation of the line that passes through the following two points. Give your answer in point-slope form and slope-intercept form.

$$(-2,4)\,,\quad(1,10)$$

Step 1

We'll need the slope of the line in order to write down the equation of the line. We'll let the first point listed above be the point (x_1, y_1) and the second point listed be the point (x_2, y_2) in the slope formula. Note that it doesn't really matter which point is which. All that matters is that you stay consistent when you plug values into the formula.

Here's the slope.

$$m = \frac{10-4}{1-(-2)} = \frac{6}{3} = \boxed{2}$$

Step 2

We'll use the point-slope form to write down the equation of the line. This requires a single point and we can use either of the points from the problem statement.

Either will give an acceptable answer here. We'll give both possible answers for the point-slope form.

$$(-2,4) \quad : \quad y = 4 + 2\left(x-(-2)\right) \quad \Rightarrow \quad \boxed{y = 4 + 2\left(x+2\right)}$$

$$(1,10) \quad : \quad \boxed{y = 10 + 2\left(x-1\right)}$$

Step 3

To get the answer in slope-intercept form all we need to do is take one of the answers from Step 2 and distribute the slope through the parenthesis and simplify. You will get the same answer regardless of which one you chose to use.

Doing this gives,

$$\boxed{y = 2x + 8}$$

4. Write down the equation of the line that passes through the following two points. Give your answer in point-slope form and slope-intercept form.

$$(8,2)\,,\quad(14,-7)$$

Step 1

We'll need the slope of the line in order to write down the equation of the line. We'll let the first point listed above be the point (x_1, y_1) and the second point listed be the point (x_2, y_2) in the slope formula. Note that it doesn't really matter which point is which. All that matters is that you stay consistent when you plug values into the formula.

Here's the slope.

$$m = \frac{-7-2}{14-8} = \frac{-9}{6} = \boxed{-\frac{3}{2}}$$

Step 2

We'll use the point-slope form to write down the equation of the line. This requires a single point and we can use either of the points from the problem statement.

Either will give an acceptable answer here. We'll give both possible answers for the point-slope form.

$$(14,-7) \quad : \quad \boxed{y = -7 - \frac{3}{2}(x-14)}$$

$$(8,2) \quad : \quad \boxed{y = 2 - \frac{3}{2}(x-8)}$$

Step 3

To get the answer in slope-intercept form all we need to do is take one of the answers from Step 2 and distribute the slope through the parenthesis and simplify. You will get the same answer regardless of which one you chose to use.

Doing this gives,

$$\boxed{y = -\frac{3}{2}x + 14}$$

---

5. Write down the equation of the line that passes through the following two points. Give your answer in point-slope form and slope-intercept form.

$$(-4,8), \quad (-1,-20)$$

Step 1

We'll need the slope of the line in order to write down the equation of the line. We'll let the first point listed above be the point (x_1, y_1) and the second point listed be the point (x_2, y_2) in the slope formula. Note that it doesn't really matter which point is which. All that matters is that you stay consistent when you plug values into the formula.

Here's the slope.

$$m = \frac{-20-8}{-1-(-4)} = \frac{-28}{3} = \boxed{-\frac{28}{3}}$$

Step 2

We'll use the point-slope form to write down the equation of the line. This requires a single point and we can use either of the points from the problem statement.

Either will give an acceptable answer here. We'll give both possible answers for the point-slope form.

$$(-4,8) \quad : \qquad y = 8 - \frac{28}{3}(x-(-4)) \qquad \Rightarrow \qquad \boxed{y = 8 - \frac{28}{3}(x+4)}$$

$$(-1,-20) \quad : \qquad y = -20 - \frac{28}{3}(x-(-1)) \qquad \Rightarrow \qquad \boxed{y = -20 - \frac{28}{3}(x+1)}$$

Step 3

To get the answer in slope-intercept form all we need to do is take one of the answers from Step 2 and distribute the slope through the parenthesis and simplify. You will get the same answer regardless of which one you chose to use.

Doing this gives,

$$\boxed{y = -\frac{28}{3}x - \frac{88}{3}}$$

---

6. Determine the slope of the line and sketch the graph of the following line.

$$4y + x = 8$$

Step 1

The first thing that we should do here is write the equation of the line in slope-intercept form. This will help with both finding the slope and with sketching the graph.

Here is the slope-intercept form of the line.

$$4y = -x + 8 \qquad \Rightarrow \qquad y = -\frac{1}{4}x + 2$$

Step 2

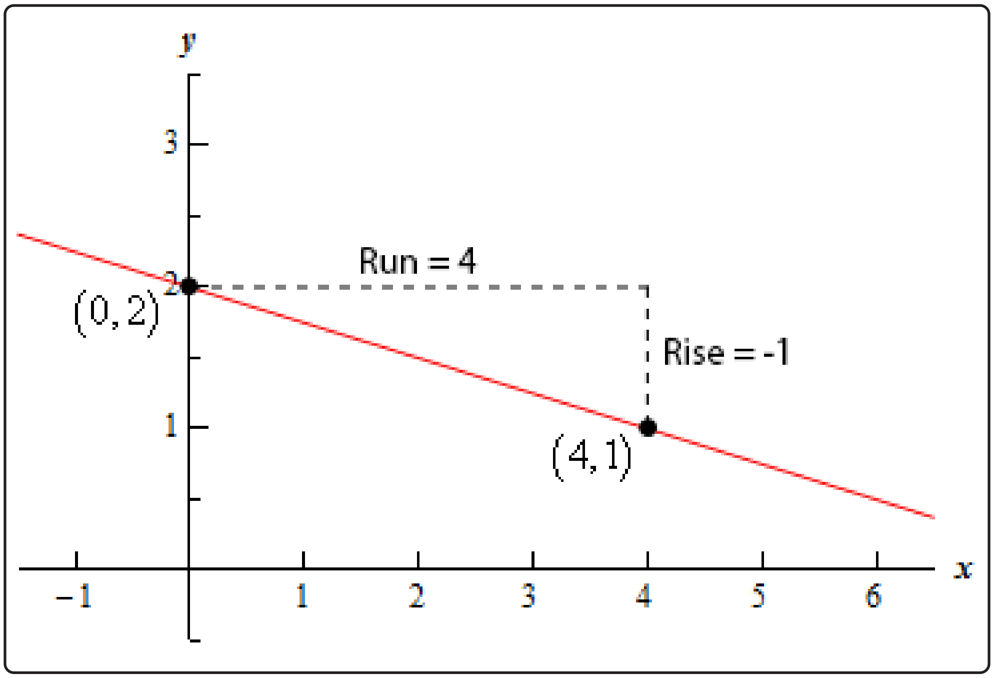
From the equation of the line in slope-intercept form that we found in the previous step we see that the slope is : $\boxed{-\frac{1}{4}}$.

Step 3

From the equation of line in slope-intercept form that we found in Step 1 we see that the y-intercept from of the line is $(0,2)$. Also, from the slope we found in Step 2 we know that the “rise” is -1 and the “run” is 4 and so a second point on the graph of the line is,

$$x_2 = 0 + 4 = 4 \qquad y_2 = 2 + (-1) = 1 \qquad \Rightarrow \qquad (4,1)$$

Using these two points we can sketch the graph of the line.



7. Determine the slope of the line and sketch the graph of the following line.

$$5x - 2y = 6$$

Step 1

The first thing that we should do here is write the equation of the line in slope-intercept form. This will help with both finding the slope and with sketching the graph.

Here is the slope-intercept form of the line.

$$\begin{aligned} 5x - 2y &= 6 \\ 5x &= 2y + 6 \\ 5x - 6 &= 2y \qquad \Rightarrow \qquad y = \frac{5}{2}x - 3 \end{aligned}$$

Step 2

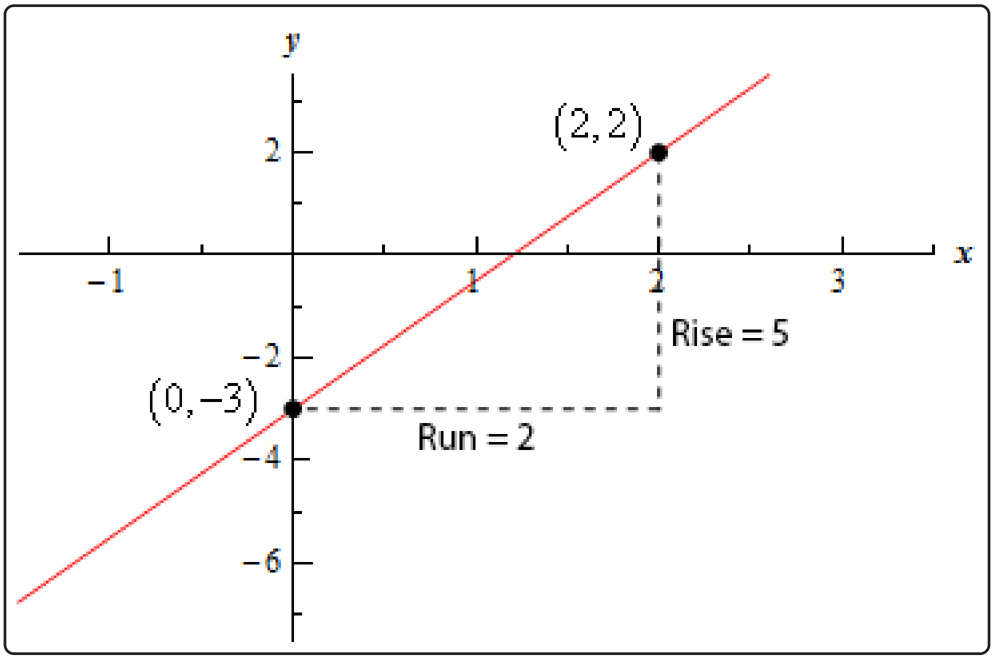
From the equation of the line in slope-intercept form that we found in the previous step we see that the slope is : $\boxed{\frac{5}{2}}$.

Step 3

From the equation of line in slope-intercept form that we found in Step 1 we see that the y-intercept from of the line is $(0,-3)$. Also, from the slope we found in Step 2 we know that the "rise" is 5 and the "run" is 2 and so a second point on the graph of the line is,

$$x_2 = 0 + 2 + 2 \qquad y_2 = -3 + 5 = 2 \qquad \Rightarrow \qquad (2,2)$$

Using these two points we can sketch the graph of the line.



8. Determine if the two lines $y = \frac{3}{7}x + 1$ and $3y + 7x = -10$ are parallel, perpendicular or neither.

Step 1

To answer this question we'll need the slope of each of the lines. The first line is in slope-intercept form and so we can easily identify the slope of that line.

$$y = \frac{3}{7}x + 1 \qquad : \qquad m_1 = \frac{3}{7}$$

For the second line let's put the equation in slope-intercept form and get its slope.

$$\begin{aligned} 3y + 7x &= -10 \\ 3y &= -7x - 10 \\ y &= -\frac{7}{3}x - \frac{10}{3} \qquad : \qquad m_2 = -\frac{7}{3} \end{aligned}$$

Step 2

The two slopes we found in the previous step are clearly not the same and so the two lines are not parallel.

On the other hand, we can see that,

$$m_1 m_2 = \left(\frac{3}{7}\right)\left(-\frac{7}{3}\right) = -1$$

and so the two lines are **perpendicular**.

9. Determine if the line $8x - y = 2$ and the line containing the two points $(1, 3)$ and $(2, -4)$ are parallel, perpendicular or neither.

Step 1

To answer this question we'll need the slope of each of the lines. For the first line let's put the equation in slope-intercept form and get its slope.

$$\begin{aligned} 8x - y &= 2 \\ y &= 8x - 2 \qquad : \qquad m_1 = 8 \end{aligned}$$

For the second line we can compute the slope directly from the two points.

$$m_2 = \frac{-4 - 3}{2 - 1} = \frac{-7}{1} = -7$$

Step 2

The two slopes we found in the previous step are clearly not the same and so the two lines are not parallel. Also, we can see that $m_1 m_2 = -56 \neq -1$ and so the lines are also not perpendicular.

Therefore, the two lines are **neither parallel nor perpendicular**.

10. Find the equation of the line through $(-7, 2)$ and is parallel to the line $3x - 14y = 4$.

Step 1

First, we need to get the slope of our new line, *i.e.* the line through the point $(-7, 2)$. We know this line is parallel to the line $3x - 14y = 4$ and so must have the same slope as the second line.

Therefore, all we need to do is put the equation of the second line into slope-intercept form and get its slope.

$$\begin{aligned} 3x - 14y &= 4 \\ 14y &= 3x - 4 \\ y &= \frac{3}{14}x - \frac{2}{7} \qquad : \qquad m_2 = \frac{3}{14} \end{aligned}$$

So, the new line must have slope of $m_1 = m_2 = \frac{3}{14}$.

Step 2

Now, we have both the slope of the new line as well as a point through the new line so we can use the point-slope form of the line to write down the equation of the new line.

$$y = 2 + \frac{3}{14}(x - (-7)) = 2 + \frac{3}{14}(x + 7) = \frac{3}{14}x + \frac{7}{2}$$

11. Find the equation of the line through $(-7, 2)$ and is perpendicular to the line $3x - 14y = 4$.

Step 1

First, we need to get the slope of our new line, *i.e.* the line through the point $(-7, 2)$. We know this line is parallel to the line $3x-14y = 4$ and the slopes must be negative reciprocals of each other.

Therefore, all we need to do is put the equation of the second line into slope-intercept form and get its slope.

$$\begin{aligned} 3x - 14y &= 4 \\ 14y &= 3x - 4 \\ y &= \frac{3}{14}x - \frac{2}{7} \qquad : \qquad m_2 = \frac{3}{14} \end{aligned}$$

So, the new line must have slope of,

$$m_1 = -\frac{1}{m_2} = -\frac{1}{{}^3/_{14}} = -\frac{14}{3}$$

Step 2

Now, we have both the slope of the new line as well as a point through the new line so we can use the point-slope form of the line to write down the equation of the new line.

$$y = 2 - \frac{14}{3}(x - (-7)) = 2 - \frac{14}{3}(x + 7) = -\frac{14}{3}x - \frac{92}{3}$$

3.3 Circles

1. Write the equation of the circle with radius 3 and center $(6,0)$.

Solution

There really isn't all that much to do with this problem. We have the radius and the center so all we need to do is plug these into the standard form of the equation from the notes.

The equation of this circle is,

$$(x-6)^2+(y-0)^2=3^2 \qquad \Rightarrow \qquad \boxed{(x-6)^2+y^2=9}$$

---

2. Write the equation of the circle with radius $\sqrt{7}$ and center $(-1,-9)$.

Solution

There really isn't all that much to do with this problem. We have the radius and the center so all we need to do is plug these into the standard form of the equation from the notes.

The equation of this circle is,

$$(x-(-1))^2+(y-(-9))^2=\left(\sqrt{7}\right)^2 \qquad \Rightarrow \qquad \boxed{(x+1)^2+(y+9)^2=7}$$

---

3. Determine the radius and center of the following circle and sketch the graph of the circle.

$$(x-9)^2+(y+4)^2=25$$

Step 1

There really isn't much to this problem. All we need to do is compare the equation we've been given to the standard form of the circle to determine the radius and center of the circle. For reference purposes here is the standard form of the circle.

$$(x-h)^2+(y-k)^2=r^2$$

So, from the first term we can quickly see that $h=9$.

Next, noticing that we can write the second term as,

$$(y+4)^2=(y-(-4))^2$$

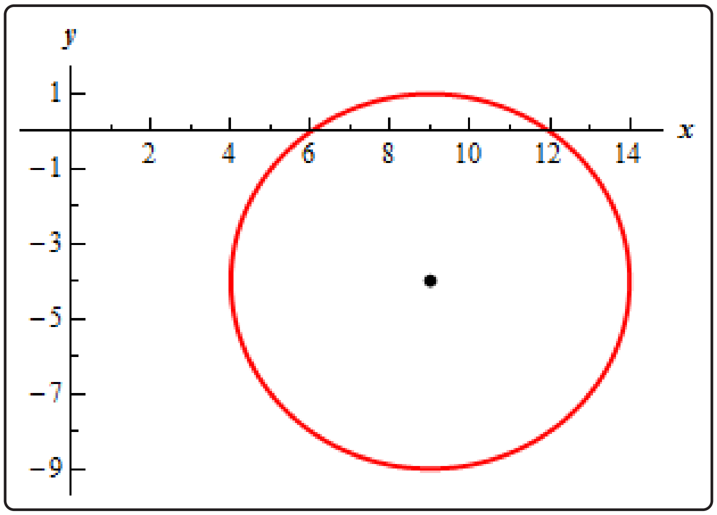
we can see that $k=-4$.

Finally, we see that $r^2 = 25$ and so we must have $r = 5$.

Therefore, the radius of the circle is 5 and the center of the circle is $(9, -4)$.

Step 2

Here is a sketch of the circle.



Remember that we can quickly graph the circle by starting at the center and then going right/left by the radius (5 for this problem) to get the right most/left most points. To get the top most/bottom most point all we need to do is start at the center and go up/down by the radius (again, 5 for this problem).

4. Determine the radius and center of the following circle and sketch the graph of the circle.

$$x^2 + (y-5)^2 = 4$$

Step 1

There really isn't much to this problem. All we need to do is compare the equation we've been given to the standard form of the circle to determine the radius and center of the circle. For reference purposes here is the standard form of the circle.

$$(x-h)^2 + (y-k)^2 = r^2$$

First, notice that we can write the first term as,

$$x^2 = (x-0)^2$$

and so we can now see that $h = 0$

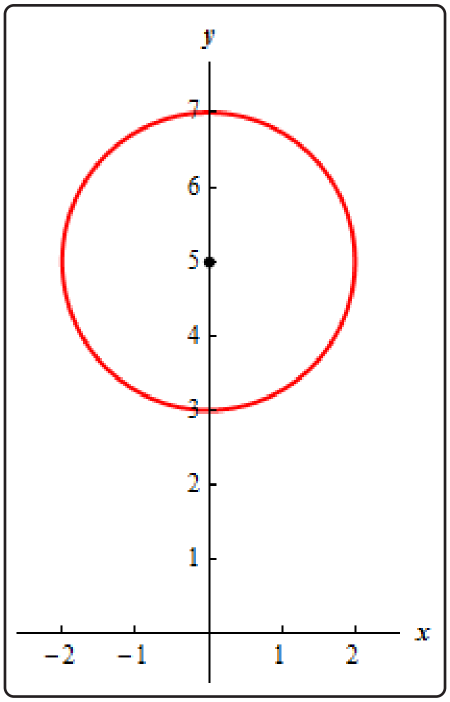
Next, from the second term we can quickly see that $k = 5$.

Finally, we see that $r^2 = 4$ and so we must have $r = 2$.

Therefore, the radius of the circle is 2 and the center of the circle is $(0, 5)$.

Step 2

Here is a sketch of the circle.



Remember that we can quickly graph the circle by starting at the center and then going right/left by the radius (2 for this problem) to get the right most/left most points. To get the top most/bottom most point all we need to do is start at the center and go up/down by the radius (again, 2 for this problem).

5. Determine the radius and center of the following circle and sketch the graph of the circle.

$$(x+1)^2 + (y+3)^2 = 6$$

Step 1

There really isn't much to this problem. All we need to do is compare the equation we've been given to the standard form of the circle to determine the radius and center of the circle. For reference purposes here is the standard form of the circle.

$$(x-h)^2+(y-k)^2=r^2$$

First, notice that we can write the first term as,

$$(x+1)^2=(x-(-1))^2$$

and so we can now see that $h=-1$

Next, we can write the second term as,

$$(y+3)^2=(y-(-3))^2$$

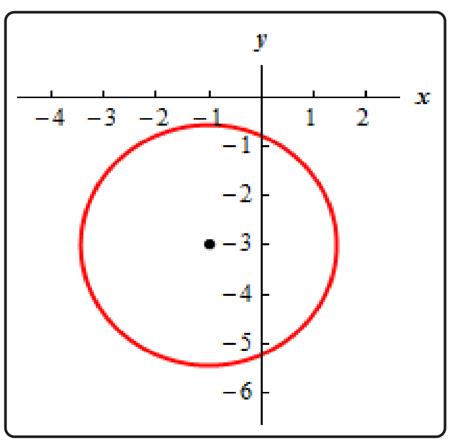
and so we can see that $k=-3$.

Finally, we see that $r^2=6$ and so we must have $r=\sqrt{6}$. Do not get excited about the fact that the radius is not an integer! There is no reason to expect the radius of a circle to be an integer and the often won't!

Therefore, the radius of the circle is $\sqrt{6}$ and the center of the circle is $(-1,-3)$.

Step 2

Here is a sketch of the circle.



Remember that we can quickly graph the circle by starting at the center and then going right/left by the radius ($\sqrt{6}\approx 2.45$ for this problem) to get the right most/left most points. To get the top most/bottom most point all we need to do is start at the center and go up/down by the radius (again, $\sqrt{6} \approx 2.45$ for this problem).

Also note that because the radius was not an integer for this problem that also means the right most/left most/top most/bottom most points will have decimal coordinates. That is okay. When sketching the graph just do your best to get the point in the approximate place and you should be fine!

---

6. Determine the radius and center of the following circle. If the equation is not the equation of a circle clearly explain why not.

$$x^2 + y^2 + 14x - 8y + 56 = 0$$

Step 1

To do this problem we need to complete the square on the x and y terms. To help with this we'll first get the number on the right side and group the x and y terms as follows.

$$x^2 + 14x + y^2 - 8y = -56$$

Step 2

Here are the numbers we need to complete the square for both x and y.

$$x: \left(\frac{14}{2}\right)^2 = (7)^2 = 49 \qquad y: \left(\frac{-8}{2}\right)^2 = (-4)^2 = 16$$

Step 3

Now, complete the square.

$$\begin{gathered} x^2 + 14x + 49 + y^2 - 8y + 16 = -56 + 49 + 16 \\ (x+7)^2 + (y-4)^2 = 9 \end{gathered}$$

Don't forget to add the numbers from Step 2 to both sides of the equation!

Step 4

So, we have the equation in standard form and so we can quickly identify the radius and center of the circle.

Radius : $r = 3$ Center : $(-7, 4)$

If you don't recall how to get the radius and center from the standard form of a circle check out Problems 3 - 5 in this section for some practice.

---

7. Determine the radius and center of the following circle. If the equation is not the equation of a circle clearly explain why not.

$$9x^2 + 9y^2 - 6x - 36y - 107 = 0$$

Step 1

To do this problem we need to complete the square on the x and y terms. To help with this we'll first get the number on the right side and group the x and y terms as follows.

$$9x^2 - 6x + 9y^2 - 36y = 107$$

Also, we know that we want the coefficients of the x^2 and y^2 to be one for the completing the square process so it will be just as easy to divide everything by a 9 to help out with the completing the square.

$$x^2 - \frac{2}{3}x + y^2 - 4y = \frac{107}{9}$$

Step 2

Here are the numbers we need to complete the square for both x and y.

$$x : \left(\frac{-{}^{2}/_{3}}{2}\right)^2 = \left(-\frac{1}{3}\right)^2 = \frac{1}{9} \qquad y : \left(\frac{-4}{2}\right)^2 = (-2)^2 = 4$$

Step 3

Now, complete the square.

$$x^2 - \frac{2}{3}x + \frac{1}{9} + y^2 - 4y + 4 = \frac{107}{9} + \frac{1}{9} + 4$$

$$\left(x - \frac{1}{3}\right)^2 + (y - 2)^2 = 16$$

Don't forget to add the numbers from Step 2 to both sides of the equation!

Step 4

So, we have the equation in standard form and so we can quickly identify the radius and center of the circle.

$$\text{Radius : } r = 4 \qquad \text{Center : } \left(\frac{1}{3}, 2\right)$$

If you don't recall how to get the radius and center from the standard form of a circle check out Problems 3 - 5 in this section for some practice.

---

8. Determine the radius and center of the following circle. If the equation is not the equation of a circle clearly explain why not.

$$x^2 + y^2 + 8x + 20 = 0$$

Step 1

To do this problem we need to complete the square on the x and y terms. To help with this we'll first get the number on the right side and group the x and y terms as follows.

$$x^2 + 8x + y^2 = -20$$

Step 2

Here is the number we need to complete the square for both x. Note that we don't need to complete the square for the y.

$$x : \left(\frac{8}{2}\right)^2 = (4)^2 = 16$$

Step 3

Now, complete the square.

$$\begin{aligned} x^2 + 8x + 16 + y^2 &= -20 + 16 \\ (x+4)^2 + y^2 &= -4 \end{aligned}$$

Don't forget to add the number from Step 2 to both sides of the equation!

Step 4

Okay, at this point we can see that this equation is not the equation of a circle. The standard form of the circle is,

$$(x-h)^2+(y-k)^2=r^2$$

The right side is r^2 and that must be a positive number (and the coefficients of the x and y must also be positive) which for our equation it is not. Therefore, this is not the equation of a circle.

3.4 The Definition of a Function

1. Determine if the following relation is a function.

$$\{(2,4),(3,-7),(6,10)\}$$

Step 1

Here is the set of 1^{st} components (*i.e.* the first number in the ordered pair) and the set of the 2^{nd} components (*i.e.* the second number in the ordered pair.

1^{st} components : $\{2,3,6\}$ 2^{nd} components : $\{-7,4,10\}$

Step 2

Pick any number from the list of 1^{st} components and identify all the ordered pairs with that number as the 1^{st} component and list all the 2^{nd} components from those ordered pairs. In this case no matter which number you pick from the 1^{st} list there is exactly one number in the second list.

Therefore, this relation is a function.

2. Determine if the following relation is a function.

$$\{(-1,8),(4,-7),(-1,6),(0,0)\}$$

Step 1

Here is the set of 1^{st} components (*i.e.* the first number in the ordered pair) and the set of the 2^{nd} components (*i.e.* the second number in the ordered pair.

1^{st} components : $\{-1,0,4\}$ 2^{nd} components : $\{-7,0,6,8\}$

Step 2

Chose -1 from the list of first components. There are two ordered pairs in the relation with -1 as the first components. One has 6 as the second component and the other has 8 as the second component.

We have found a number from the 1^{st} list that has two numbers in the 2^{nd} list associated with it. Therefore, this relation is NOT a function.

3. Determine if the following relation is a function.

$$\{(2,1),(9,10),(-4,10),(-8,1)\}$$

Step 1

Here is the set of 1^{st} components (*i.e.* the first number in the ordered pair) and the set of the 2^{nd} components (*i.e.* the second number in the ordered pair.

1^{st} components : $\{-8,-4,2,9\}$ 2^{nd} components : $\{1,10\}$

Step 2

Pick any number from the list of 1^{st} components and identify all the ordered pairs with that number as the 1^{st} component and list all the 2^{nd} components from those ordered pairs. In this case no matter which number you pick from the 1^{st} list there is exactly one number in the second list.

Therefore, this relation is a function.

Note that each number in the 2^{nd} list does have two numbers associated with it from the 1^{st} list. That is not an issue however. We only have an issue if numbers from the 1^{st} list have more than one number from the 2^{nd} list associated with it.

4. Determine if the given equation is a function.

$$y = 14 - \frac{1}{3}x$$

Solution

To directly determine if an equation is a function can be quite difficult at times. What we need to do is show that for each x that we plug into the equation we can only get a single y out of the equation. For this case we can kind of talk our way through this.

Look at the equation and notice that if we were to plug any x into the equation all we would do is multiply the x by $-\frac{1}{3}$ and then add 14. For each of these algebraic operations there is exactly one number that results and so y can only be a single value regardless of the x we plug in.

Therefore, this equation is a function.

5. Determine if the given equation is a function.

$$y = \sqrt{3x^2 + 1}$$

Solution

To directly determine if an equation is a function can be quite difficult at times. What we need to do is show that for each x that we plug into the equation we can only get a single y out of the equation. For this case we can kind of talk our way through this.

Look at the equation and notice that if we were to plug any x into the equation all we would do is square the x, multiply that by 3 and then add 1 to that result. For each of these algebraic operations there is exactly one number that results.

Finally, all we do is take the square root of that result. Recalling that square roots can only give positive numbers (*i.e.* we don't add on a $\pm$ when we take the square root).

So, y can only be a single value regardless of the x we plug in.

Therefore, this equation is a function.

---

6. Determine if the given equation is a function.

$$y^4 - x^2 = 16$$

Solution

To directly determine if an equation is a function can be quite difficult at times. What we need to do is show that for each x that we plug into the equation we can only get a single y out of the equation.

For this equation let's first rewrite it a little as follows,

$$y^4 = x^2 + 16$$

Now, let's take a look at a specific x, say $x = 0$. If we plug this into the equation we get,

$$y^4 = 0^2 + 16 = 16$$

Now, at this point we can see that there are two possible y values, $y = -2$ or $y = 2$ since for both we have,

$$(-2)^4 = 16 \qquad \text{and} \qquad (2)^4 = 16$$

So, we've found an x for which the equation gives two possible y values. Note as well that, for this equation, it doesn't matter which x we choose to use we will get the same result.

Therefore, this equation is NOT a function.

---

7. Given $f(x) = 3 - 2x^2$ determine each of the following.

(a) $f(0)$ **(b)** $f(2)$ **(c)** $f(-4)$ **(d)** $f(3t)$ **(e)** $f(x+2)$

Solutions

(a) $f(0)$

Solution

Remember that for function evaluation all we need to do is replace all the x's (only one in this case) with the number that is in the parenthesis. Doing this for this part gives,

$$f(0) = 3 - 2(0)^2 = \boxed{3}$$

(b) $f(2)$

Solution

Remember that for function evaluation all we need to do is replace all the x's (only one in this case) with the number that is in the parenthesis. Doing this for this part gives,

$$f(2) = 3 - 2(2)^2 = 3 - 2(4) = \boxed{-5}$$

(c) $f(-4)$

Solution

Remember that for function evaluation all we need to do is replace all the x's (only one in this case) with the number that is in the parenthesis. Doing this for this part gives,

$$f(-4) = 3 - 2(-4)^2 = 3 - 2(16) = \boxed{-29}$$

(d) $f(3t)$

Solution

Remember that for function evaluation all we need to do is replace all the x's (only one in this case) with whatever is in the parenthesis. It doesn't matter if the stuff in the parenthesis is not a number the evaluation works exactly the same as if it was a number.

Doing the evaluation for this part gives,

$$f(3t) = 3 - 2(3t)^2 = 3 - 2\left(9t^2\right) = \boxed{3 - 18t^2}$$

(e) $f(x+2)$

Solution

Remember that for function evaluation all we need to do is replace all the x's (only one in this case) with whatever is in the parenthesis. It doesn't matter if the stuff in the parenthesis is not a number the evaluation works exactly the same as if it was a number. Do not get excited about the fact that the stuff in the parenthesis also involves an x. Again, it does not matter, the evaluation still works the same way.

Doing the evaluation for this part gives,

$$f(x+2) = 3 - 2(x+2)^2 = 3 - 2\left(x^2 + 4x + 4\right) = \boxed{-2x^2 - 8x - 5}$$

---

8. Given $g\left(w\right) = \dfrac{4}{w+1}$ determine each of the following.

(a) $g\left(-6\right)$ **(b)** $g\left(-2\right)$ **(c)** $g\left(0\right)$ **(d)** $g\left(t-1\right)$ **(e)** $g\left(4w+3\right)$

Solutions

(a) $g\left(-6\right)$

Solution

Remember that for function evaluation all we need to do is replace all the w's (only one in this case) with the number that is in the parenthesis. Doing this for this part gives,

$$g\left(-6\right) = \frac{4}{-6+1} = \boxed{-\frac{4}{5}}$$

(b) $g\left(-2\right)$

Solution

Remember that for function evaluation all we need to do is replace all the w's (only one in this case) with the number that is in the parenthesis. Doing this for this part gives,

$$g\left(-2\right) = \frac{4}{-2+1} = \boxed{-4}$$

(c) $g\left(0\right)$

Solution

Remember that for function evaluation all we need to do is replace all the w's (only one in this case) with the number that is in the parenthesis. Doing this for this part gives,

$$g\left(0\right) = \frac{4}{0+1} = \boxed{4}$$

(d) $g\left(t-1\right)$

Solution

Remember that for function evaluation all we need to do is replace all the w's (only one in this case) with whatever is in the parenthesis. It doesn't matter if the stuff in the parenthesis is not a number the evaluation works exactly the same as if it was a number.

Doing the evaluation for this part gives,

$$g\left(t-1\right)=\frac{4}{\left(t-1\right)+1}=\boxed{\frac{4}{t}}$$

(e) $g\left(4w+3\right)$

Solution

Remember that for function evaluation all we need to do is replace all the w's (only one in this case) with whatever is in the parenthesis. It doesn't matter if the stuff in the parenthesis is not a number the evaluation works exactly the same as if it was a number. Do not get excited about the fact that the stuff in the parenthesis also involves an w. Again, it does not matter, the evaluation still works the same way.

Doing the evaluation for this part gives,

$$g\left(4w+3\right)=\frac{4}{\left(4w+3\right)+1}=\frac{4}{4w+4}=\boxed{\frac{1}{w+1}}$$

---

9. Given $h(t) = t^2 + 6$ determine each of the following.

(a) $h(0)$ **(b)** $h(-2)$ **(c)** $h(2)$ **(d)** $h(\sqrt{x})$ **(e)** $h(3-t)$

Solutions

(a) $h(0)$

Solution

Remember that for function evaluation all we need to do is replace all the t's (only one in this case) with the number that is in the parenthesis. Doing this for this part gives,

$$h(0) = (0)^2 + 6 = \boxed{6}$$

(b) $h(-2)$

Solution

Remember that for function evaluation all we need to do is replace all the t's (only one in this case) with the number that is in the parenthesis. Doing this for this part gives,

$$h(-2) = (-2)^2 + 6 = 4 + 6 = \boxed{10}$$

(c) $h(2)$

Solution

Remember that for function evaluation all we need to do is replace all the t's (only one in this case) with the number that is in the parenthesis. Doing this for this part gives,

$$h(2) = (2)^2 + 6 = 4 + 6 = \boxed{10}$$

(d) $h\left(\sqrt{x}\right)$

Solution

Remember that for function evaluation all we need to do is replace all the t's (only one in this case) with whatever is in the parenthesis. It doesn't matter if the stuff in the parenthesis is not a number the evaluation works exactly the same as if it was a number.

Doing the evaluation for this part gives,

$$h\left(\sqrt{x}\right) = \left(\sqrt{x}\right)^2 + 6 = \boxed{x+6}$$

(e) $h\left(3-t\right)$

Solution

Remember that for function evaluation all we need to do is replace all the t's (only one in this case) with whatever is in the parenthesis. It doesn't matter if the stuff in the parenthesis is not a number the evaluation works exactly the same as if it was a number. Do not get excited about the fact that the stuff in the parenthesis also involves an t. Again, it does not matter, the evaluation still works the same way.

Doing the evaluation for this part gives,

$$h\left(3-t\right) = \left(3-t\right)^2 + 6 = 9 - 6t + t^2 + 6 = \boxed{t^2 - 6t + 15}$$

---

10. Given $h(z) = \begin{cases} 3z & \text{if } z < 2 \\ 1 + z^2 & \text{if } z \geq 2 \end{cases}$ determine each of the following.

(a) $h(0)$ **(b)** $h(2)$ **(c)** $h(7)$

Solutions

(a) $h(0)$

Solution

Remember that for piecewise functions we use the equation for which the number in the parenthesis meets the condition.

For this problem we can see that $0 < 2$ and so we use top equation to do the evaluation.

$$h(0) = 3(0) = \boxed{0}$$

(b) $h(2)$

Solution

Remember that for piecewise functions we use the equation for which the number in the parenthesis meets the condition.

For this problem we can see that $2 \geq 2$ and so we use bottom equation to do the evaluation.

$$h(2) = 1 + (2)^2 = \boxed{5}$$

(c) $h(7)$

Solution

Remember that for piecewise functions we use the equation for which the number in the parenthesis meets the condition.

For this problem we can see that $7 \geq 2$ and so we use bottom equation to do the evaluation.

$$h(7) = 1 + (7)^2 = \boxed{50}$$

11. Given $f\left(x\right) = \begin{cases} 6 & \text{if } x \geq 9 \\ x+9 & \text{if } 2 < x < 9 \\ x^2 & \text{if } x \leq 2 \end{cases}$ determine each of the following.

(a) $f\left(-4\right)$ **(b)** $f\left(2\right)$ **(c)** $f\left(6\right)$ **(d)** $f\left(9\right)$ **(e)** $f\left(12\right)$

Solutions

(a) $f\left(-4\right)$

Solution

Remember that for piecewise functions we use the equation for which the number in the parenthesis meets the condition.

For this problem we can see that $-4 \leq 2$ and so we use bottom equation to do the evaluation.

$$f\left(-4\right) = \left(-4\right)^2 = \boxed{16}$$

(b) $f\left(2\right)$

Solution

Remember that for piecewise functions we use the equation for which the number in the parenthesis meets the condition.

For this problem we can see that $2 \leq 2$ and so we use bottom equation to do the evaluation.

$$f\left(2\right) = \left(2\right)^2 = \boxed{4}$$

(c) $f\left(6\right)$

Solution

Remember that for piecewise functions we use the equation for which the number in the parenthesis meets the condition.

For this problem we can see that $2 < 6 < 9$ and so we use middle equation to do the evaluation.

$$f\left(6\right) = 6 + 9 = \boxed{15}$$

(d) $f(9)$

Solution

Remember that for piecewise functions we use the equation for which the number in the parenthesis meets the condition.

For this problem we can see that $9 \geq 9$ and so we use top equation to do the evaluation.

$$f(9) = \boxed{6}$$

Do not get excited about the fact that there are no x's in the equation we are evaluating!

(e) $f(12)$

Solution

Remember that for piecewise functions we use the equation for which the number in the parenthesis meets the condition.

For this problem we can see that $12 \geq 9$ and so we use top equation to do the evaluation.

$$f(12) = \boxed{6}$$

Do not get excited about the fact that there are no x's in the equation we are evaluating!

---

12. The difference quotient for the function $f(x)$ is defined to be,

$$\frac{f(x+h)-f(x)}{h}$$

Compute the difference quotient for the function $f(x) = 4 - 9x$.

Step 1

We'll work this problem in parts. First let's compute $f(x+h)$.

$$f(x+h) = 4 - 9(x+h) = 4 - 9x - 9h$$

Step 2

Now we'll compute $f(x+h) - f(x)$ and do a little simplification.

$$f(x+h) - f(x) = 4 - 9x - 9h - (4 - 9x) = 4 - 9x - 9h - 4 + 9x = -9h$$

Be careful with the parenthesis when subtracting $f(x)$. We need to subtract the function and so we need parenthesis around the whole thing to make sure we do subtract the function.

Step 3

We can now finish the problem by computing the full difference quotient.

$$\frac{f(x+h)-f(x)}{h} = \frac{-9h}{h} = \boxed{-9}$$

13. The difference quotient for the function $f(x)$ is defined to be,

$$\frac{f(x+h)-f(x)}{h}$$

Compute the difference quotient for the function $f(x) = 2x^2 - x$.

Step 1

We'll work this problem in parts. First let's compute $f(x+h)$.

$$\begin{aligned} f(x+h) &= 2(x+h)^2 - (x+h) = 2\left(x^2 + 2xh + h^2\right) - (x+h) \\ &= \boxed{2x^2 + 4xh + 2h^2 - x - h} \end{aligned}$$

Step 2

Now we'll compute $f(x+h) - f(x)$ and do a little simplification.

$$\begin{aligned} f(x+h) - f(x) &= 2x^2 + 4xh + 2h^2 - x - h - \left(2x^2 - x\right) \\ &= 2x^2 + 4xh + 2h^2 - x - h - 2x^2 + x = \boxed{4xh + 2h^2 - h} \end{aligned}$$

Be careful with the parenthesis when subtracting $f(x)$. We need to subtract the function and so we need parenthesis around the whole thing to make sure we do subtract the function.

Step 3

We can now finish the problem by computing the full difference quotient.

$$\frac{f(x+h)-f(x)}{h} = \frac{4xh + 2h^2 - h}{h} = \frac{h(4x + 2h - 1)}{h} = \boxed{4x + 2h - 1}$$

14. Determine the domain of the following function.

$$A(x) = 6x + 14$$

Solution

Recall that the domain is simply all the values of x that we can plug into the function and the resulting function value will be a real number.

In this case we can see that no matter what value of x we plug into the function 6 times that number plus 14 will be a real number and so for this function the domain is **all real numbers**.

---

15. Determine the domain of the following function.

$$f(x) = \frac{1}{x^2 - 25}$$

Solution

Recall that the domain is simply all the values of x that we can plug into the function and the resulting function value will be a real number.

In this case we have a rational expression where the numerator is a constant (and so won't cause any problems for any value of x) and the denominator is a polynomial.

So, we need to avoid division by zero for this problem. We will get division by zero at,

$$x^2 - 25 = 0 \qquad \to \quad x^2 = 25 \qquad \to \quad x = \pm 5$$

Therefore, the domain for this function is **all real numbers except** $x = \pm 5$ **.**

---

16. Determine the domain of the following function.

$$g(t) = \frac{8t - 24}{t^2 - 7t - 18}$$

Solution

Recall that the domain is simply all the values of t that we can plug into the function and the resulting function value will be a real number.

In this case we have a rational expression where both the numerator and denominator are polynomials.

The numerator won't cause any problems since we can plug any value of t into the numerator and get back a real number.

So, for this problem, all we need to avoid is division by zero. We will get division by zero at,

$$t^2 - 7t - 18 = 0 \quad \rightarrow \quad (t-9)(t+2) = 0 \quad \rightarrow \quad t = -2, \quad t = 9$$

Therefore, the domain for this function is **all real numbers except** $t = -2$ **and** $t = 9$ **.**

---

17. Determine the domain of the following function.

$$g(w) = \sqrt{9w+7}$$

Solution

Recall that the domain is simply all the values of w that we can plug into the function and the resulting function value will be a real number.

In this case we have a square root and so we know that we need to require that the quantity under the root be positive or zero.

In other words, we need to require,

$$\begin{aligned} 9w + 7 &\geq 0 \\ 9w &\geq -7 \\ w &\geq -\frac{7}{9} \end{aligned}$$

Therefore, the domain for this function is $w \geq -\frac{7}{9}$ **.**

---

18. Determine the domain of the following function.

$$f(x) = \frac{1}{\sqrt{x^2 - 8x + 15}}$$

Solution

Recall that the domain is simply all the values of x that we can plug into the function and the resulting function value will be a real number.

In this case we have a square root in the denominator of a rational expression. The numerator is a constant and so won't cause any problems for us. However, because of the root in the denominator we need to require that the quantity under the root be positive.

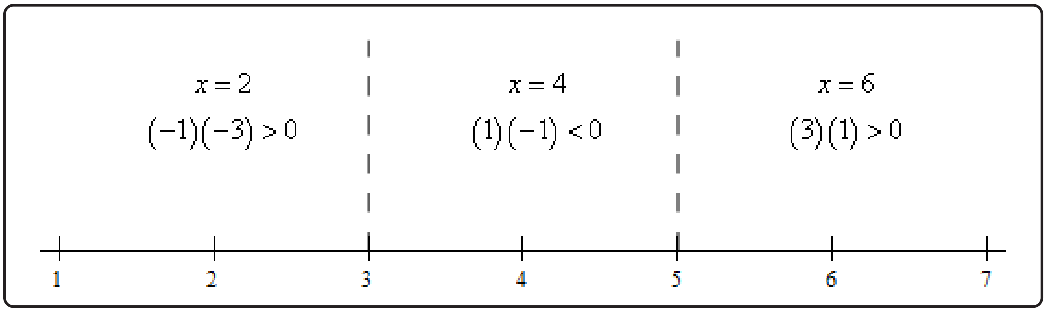
Note as well that while we don't have a problem with zero under a square root because the root is in the denominator allowing the quantity under the root to be zero would result in a division by zero problem and so we need to also avoid that.

In other words, we need to require,

$$\begin{aligned} x^2 - 8x + 15 &> 0 \\ (x-3)(x-5) &> 0 \end{aligned}$$

If you don't recall how to solve polynomial inequalities you should go back to that section and do some of the practice problems there. Here is a quick solution for this one.

We can see that the polynomial on the left is zero at $x = 3$ and $x = 5$. A quick number line for this inequality is,



From this number line we can see that the polynomial is positive in the ranges $x < 3$ and $x > 5$.

Therefore, the domain for this function is $x < 3$ and $x > 5$.

3.5 Graphing Functions

1. Construct a table of at least 4 ordered pairs of points on the graph of the following function and use the ordered pairs from the table to sketch the graph of the function.

$$f(x) = x^2 - 2$$

Hint

By this time we've seen enough graphs of equations that we should be getting a feel for points to choose. However, if haven't gotten a feel it yet try a selection of at least a couple of points to either side of zero (maybe even including zero).

Step 1

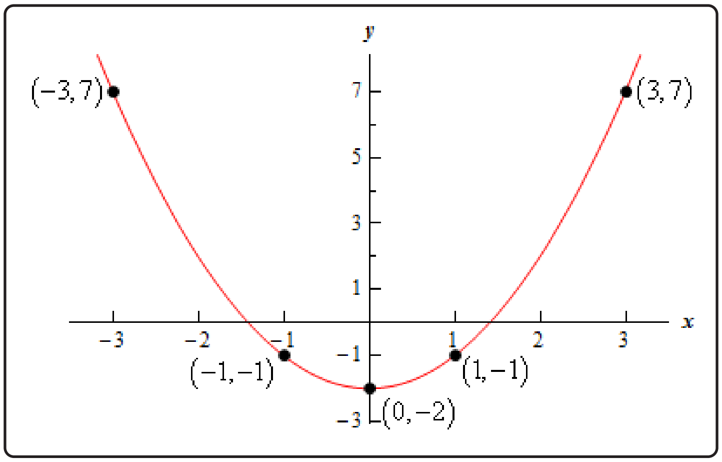
For this function a selection of points on either side of zero should be sufficient to get an idea of what the graph of this function looks like. We'll also include zero for no other reason that it will give an extra point on the graph.

Here is the table of points we'll use for this problem.

x	$f(x)$	(x,y)
-3	7	$(-3,7)$
-1	-1	$(-1,-1)$
0	-2	$(0,-2)$
1	-1	$(1,-1)$
3	7	$(3,7)$

Step 2

Here is a sketch of the function.



2. Construct a table of at least 4 ordered pairs of points on the graph of the following function and use the ordered pairs from the table to sketch the graph of the function.

$$f(x) = \sqrt{x+1}$$

Hint

By this time we've seen enough graphs of equations that we should be getting a feel for points to choose. However, if haven't gotten a feel it yet try a selection points which will give function values that are easy to graph, *i.e.* function values that are integers.

Step 1

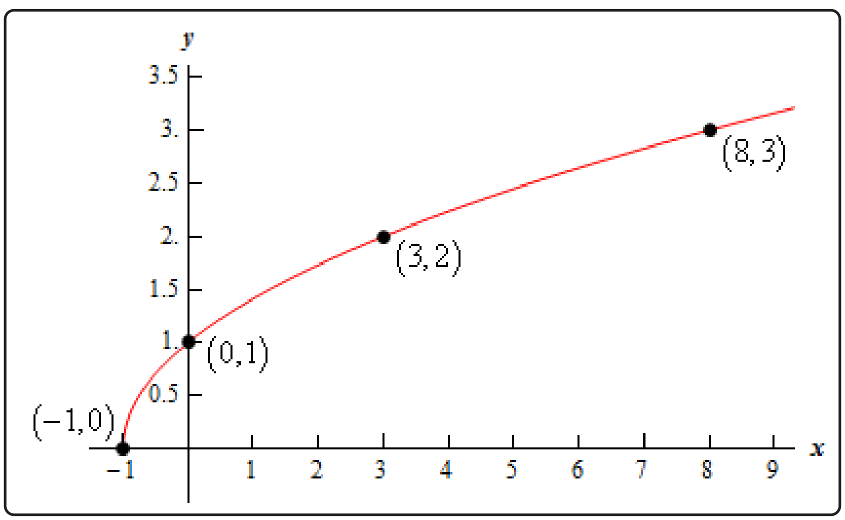
For this function we'll use points that give function values that are integers to make the graphing a little easier.

Here is the table of points we'll use for this problem.

x	$f(x)$	(x,y)
-1	0	$(-1,0)$
0	1	$(0,1)$
3	2	$(3,2)$
8	3	$(8,3)$

Step 2

Here is a sketch of the function.



3. Construct a table of at least 4 ordered pairs of points on the graph of the following function and use the ordered pairs from the table to sketch the graph of the function.

$$f(x) = 9$$

Hint

Do not get excited about the lack of x's on the right side. This function evaluates just like all the functions you've evaluated.

Step 1

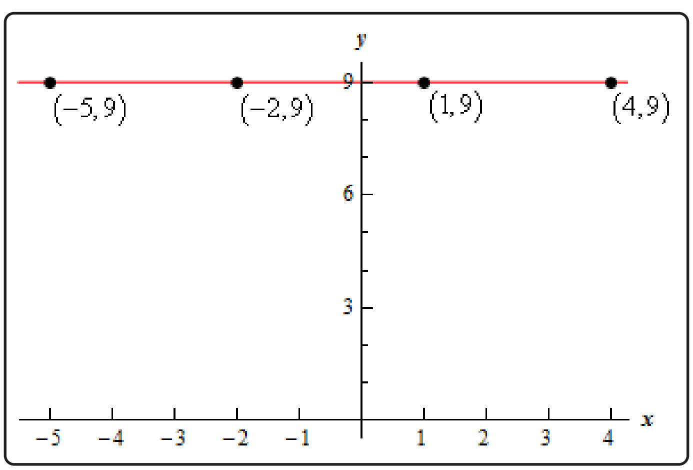
This is a <b style="mso-bidi-font-weight:normal">constant function. The fact that there are no x's on the right side is not a problem. Here is the table of points we'll use for this problem.

x	$f(x)$	(x, y)
-5	9	$(-5, 9)$
-2	9	$(-2, 9)$
1	9	$(1, 9)$
4	9	$(4, 9)$

Because there are no x's on the right side there are no x's to plug numbers in. That in turn means that no matter what x we plug into the function we always get a function value of 9 as our table above shows.

Step 2

Here is a sketch of the function.



4. Construct a table of at least 4 ordered pairs of points on the graph of the following function and use the ordered pairs from the table to sketch the graph of the function.

$$f(x) = \begin{cases} 10 - 2x & \text{if } x < 2 \\ x^2 + 2 & \text{if } x \geq 2 \end{cases}$$

Hint

Remember to pick points from both ranges and don't forget that you also need to know the value of each function at the "cutoff point".

Step 1

We have a piecewise function here so we need to make sure and pick points from each range of x when building our table of points. Also, we'll want to plug the "cutoff point", $x = 2$, into the top equation as well so we know where it will be at the end of the range of x's for which it is valid.

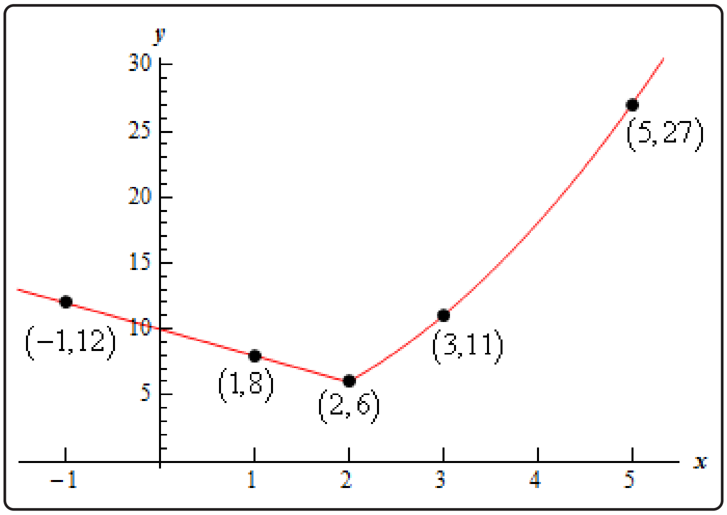
Here is the table of points we'll use for this problem.

x	$10 - 2x$	(x, y)
-1	12	$(-1, 12)$
1	8	$(1, 8)$
2	6	$(2, 6)$

x	$x^2 + 2$	(x, y)
2	6	$(2, 6)$
3	11	$(3, 11)$
5	27	$(5, 27)$

Step 2

Here is a sketch of the function.



Note that in this case the two pieces "met" at the point $(2, 6)$. Sometimes this will happen with these problems and sometimes it won't. This is the reason we always evaluate both equations at the "cutoff point". Without doing that we wouldn't know if the pieces meet or not.

---

5. Construct a table of at least 4 ordered pairs of points on the graph of the following function and use the ordered pairs from the table to sketch the graph of the function.

$$f(x) = \begin{cases} 5 + x & \text{if } x \geq 1 \\ 2 & \text{if } -2 \leq x < 1 \\ 1 - x^2 & \text{if } x < -2 \end{cases}$$

Hint

Remember to pick points from both ranges and don't forget that you also need to know the value of each function at the "cutoff points".

Step 1

We have a piecewise function here so we need to make sure and pick points from each range of x when building our table of points. Also, we'll want to plug the "cutoff points", $x = -2$ and $x = 1$, into the corresponding equation as well so we know where each will be at the end of the range of x's for which it is valid.

Here is the table of points we'll use for this problem.

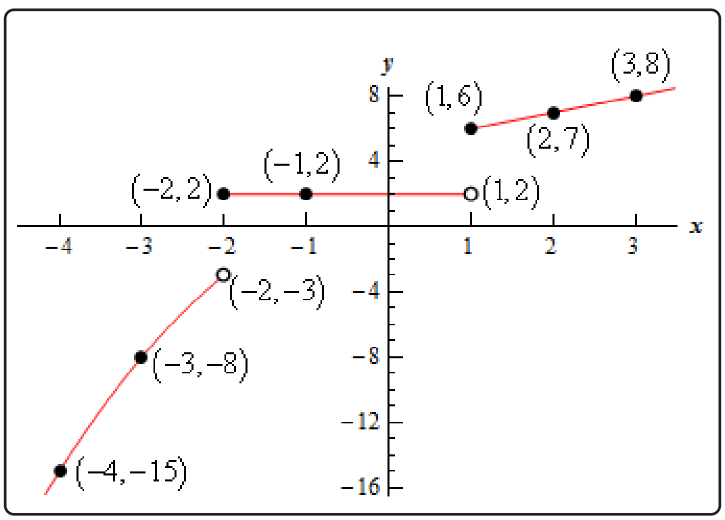
x	$1-x^2$	(x,y)
-4	-15	$(-4,-15)$
-3	-8	$(-3,-8)$
-2	-3	$(-2,-3)$

x	2	(x,y)
-2	2	$(-2,2)$
-1	2	$(-1,2)$
1	2	$(1,2)$

x	$5+x$	(x,y)
1	6	$(1,6)$
2	7	$(2,7)$
3	8	$(3,8)$

Step 2

Here is a sketch of the function.



In this case neither of the pieces of the graph met at the cutoff points. This is the reason we compute the value of each equation at the cutoff points. Without doing that we wouldn't know if they met at the cutoff point or not.

3.6 Combining Functions

1. Given $f(x) = 6x + 2$ and $g(x) = 10 - 7x$ compute each of the following.

(a) $(f - g)(2)$ **(b)** $(g - f)(2)$ **(c)** $f\,g$ **(d)** $\left(\frac{f}{g}\right)(x)$

Solutions

(a) $(f - g)(2)$

Solution

Not much to do here other than do the evaluation. We'll leave it to you to verify the specific function evaluations.

$$(f - g)(2) = f(2) - g(2) = 14 - (-4) = \boxed{18}$$

(b) $(g - f)(2)$

Solution

Not much to do here other than do the evaluation. We'll leave it to you to verify the specific function evaluations.

$$(g - f)(2) = g(2) - f(2) = -4 - 14 = \boxed{-18}$$

(c) $f\,g$

Solution

Not much to do here other than do the evaluation. Also, remember that in this case the "evaluation" really only consists of plugging the two equations for each function in and doing some basic algebraic manipulations to simplify the answer if possible.

$$f\,g = f(x)\,g(x) = (6x + 2)(10 - 7x) = \boxed{-42x^2 + 46x + 20}$$

(d) $\left(\frac{f}{g}\right)(x)$

Solution

Not much to do here other than do the evaluation. Also, remember that in this case the "evaluation" really only consists of plugging the two equations for each function in and doing some basic algebraic manipulations to simplify the answer if possible, although in this case there really isn't any simplification that can be done.

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \boxed{\frac{6x+2}{10-7x}}$$

---

2. Given $P(t) = 4t^2 + 3t - 1$ and $A(t) = 2 - t^2$ compute each of the following.

(a) $(P + A)(t)$ **(b)** $A - P$ **(c)** $(PA)(t)$ **(d)** $\left(\frac{P}{A}\right)(-2)$

Solutions

(a) $(P + A)(t)$

Solution

Not much to do here other than do the evaluation. Also, remember that in this case the "evaluation" really only consists of plugging the two equations for each function in and doing some basic algebraic manipulations to simplify the answer if possible.

$$(P + A)(t) = P(t) + A(t) = 4t^2 + 3t - 1 + 2 - t^2 = \boxed{3t^2 + 3t + 1}$$

(b) $A - P$

Solution

Not much to do here other than do the evaluation. Also, remember that in this case the "evaluation" really only consists of plugging the two equations for each function in and doing some basic algebraic manipulations to simplify the answer if possible.

$$A - P = A(t) - P(t) = 2 - t^2 - \left(4t^2 + 3t - 1\right) = 2 - t^2 - 4t^2 - 3t + 1 = \boxed{-5t^2 - 3t + 3}$$

(c) $(PA)(t)$

Solution

Not much to do here other than do the evaluation. Also, remember that in this case the "evaluation" really only consists of plugging the two equations for each function in and doing some basic algebraic manipulations to simplify the answer if possible.

$$\begin{aligned}(PA)(t) = P(t)\,A(t) &= \left(4t^2+3t-1\right)\left(2-t^2\right)\\ &= 8t^2-4t^4+6t-3t^3-2+t^2\\ &= \boxed{-4t^4-3t^3+9t^2+6t-2}\end{aligned}$$

For the product just remember to distribute every term from the first polynomial through the second polynomial and the combine like terms to simplify.

(d) $\left(\dfrac{P}{A}\right)(-2)$

Solution

Not much to do here other than do the evaluation. We'll leave it to you to verify the specific function evaluations.

$$\left(\frac{P}{A}\right)(-2) = \frac{P(-2)}{A(-2)} = \frac{9}{-2} = \boxed{-\frac{9}{2}}$$

3. Given $h(z) = 7z + 6$ and $f(z) = 4 - z$ compute each of the following.

(a) $(f\,h)(z)$ **(b)** $(f \circ h)(z)$ **(c)** $(h \circ f)(z)$ **(d)** $(h \circ h)(z)$

Solutions

(a) $(f\,h)(z)$

Solution

Not much to do here other than do the multiplication. Also, remember that in this case the "evaluation" really only consists of plugging the two equations for each function in and doing some basic algebraic manipulations to simplify the answer if possible.

$$(f\,h)(z) = f(z)\,h(z) = (7z+6)(4-z) = \boxed{-7z^2 + 22z + 24}$$

(b) $(f \circ h)(z)$

Solution

Remember that the "∘" denotes composition and not multiplication! Also remember that the order of the composition as written in the statement needs to be followed!

$$(f \circ h)(z) = f[h(z)] = f[7z+6] = 4 - (7z+6) = \boxed{-7z - 2}$$

(c) $(h \circ f)(z)$

Solution

Remember that the "∘" denotes composition and not multiplication! Also remember that the order of the composition as written in the statement needs to be followed!

$$(h \circ f)(z) = h[f(z)] = h[4-z] = 7(4-z) + 6 = \boxed{-7z + 34}$$

(d) $(h \circ h)(z)$

Solution

Remember that the "$\circ$" denotes composition and not multiplication and do not get excited about the fact that each portion is the same function. It works exactly like every other composition problem.

$$(h \circ h)(z) = h[h(z)] = h[7z+6] = 7(7z+6)+6 = \boxed{49z+48}$$

---

4. Given $f(x) = 6x^2$ and $g(x) = x^2+3x-1$ compute each of the following.

(a) $(g\,f)(x)$ **(b)** $(f \circ g)(x)$ **(c)** $(g \circ f)(x)$ **(d)** $(f \circ f)(x)$

Solutions

(a) $(g\,f)(x)$

Solution

Not much to do here other than do the multiplication. Also, remember that in this case the "evaluation" really only consists of plugging the two equations for each function in and doing some basic algebraic manipulations to simplify the answer if possible.

$$(g\,f)(x) = g(x)\,f(x) = \left(x^2+3x-1\right)\left(6x^2\right) = \boxed{6x^4+18x^3-6x^2}$$

(b) $(f \circ g)(x)$

Solution

Remember that the "$\circ$" denotes composition and not multiplication! Also remember that the order of the composition as written in the statement needs to be followed!

$$\begin{aligned}(f \circ g)(x) = f[g(x)] &= f\left[x^2+3x-1\right] \\ &= 6\left(x^2+3x-1\right)^2 \\ &= 6\left(x^4+6x^3+7x^2-6x+1\right) \\ &= \boxed{6x^4+36x^3+42x^2-36x+6}\end{aligned}$$

(c) $(g \circ f)(x)$

Solution

Remember that the "$\circ$" denotes composition and not multiplication! Also remember that the order of the composition as written in the statement needs to be followed!

$$(g \circ f)(x) = g[f(x)] = g\left[6x^2\right] = \left(6x^2\right)^2 + 3\left(6x^2\right) - 1 = \boxed{36x^4 + 18x^2 - 1}$$

(d) $(f \circ f)(x)$

Solution

Remember that the "$\circ$" denotes composition and not multiplication and do not get excited about the fact that each portion is the same function. It works exactly like every other composition problem.

$$(f \circ f)(x) = f[f(x)] = f\left[6x^2\right] = 6\left(6x^2\right)^2 = 6\left(36x^4\right) = \boxed{216x^4}$$

5. Given $R(t) = \sqrt{t} - 2$ and $A(t) = (t+2)^2$, $t \geq 0$ compute each of the following.

(a) $(R \circ A)(t)$ **(b)** $(A \circ R)(t)$

Solutions

(a) $(R \circ A)(t)$

Solution

Remember that the "$\circ$" denotes composition and not multiplication!

$$(R \circ A)(t) = R[A(t)] = R\left[(t+2)^2\right] = \sqrt{(t+2)^2} - 2 = (t+2) - 2 = \boxed{t}$$

(b) $(A \circ R)(t)$

Solution

Remember that the "$\circ$" denotes composition and not multiplication!

$$(A \circ R)(t) = A[R(t)] = A\left[\sqrt{t} - 2\right] = \left(\sqrt{t} - 2 + 2\right)^2 = \left(\sqrt{t}\right)^2 = \boxed{t}$$

3.7 Inverse Functions

1. Given $h(x) = 5 - 9x$ find $h^{-1}(x)$.

Hint

Just follow the process outlines in the notes and you'll be set to do this problem!

Step 1

For the first step we simply replace the function with a y.

$$y = 5 - 9x$$

Step 2

Next, replace all the x's with y's and all the original y's with x's.

$$x = 5 - 9y$$

Step 3

Solve the equation from Step 2 for y.

$$\begin{aligned} x &= 5 - 9y \\ 9y &= 5 - x \\ y &= \frac{5 - x}{9} \end{aligned}$$

Step 4

Replace y with $h^{-1}(x)$.

$$h^{-1}(x) = \frac{5 - x}{9}$$

Step 5

Finally, do a quick check by computing one or both of $\left(h \circ h^{-1}\right)(x)$ and $\left(h^{-1} \circ h\right)(x)$ and verify that each is x. In general, we usually just check one of these and well do that here.

$$\left(h \circ h^{-1}\right)(x) = h\left[h^{-1}(x)\right] = P\left[\frac{5-x}{9}\right] = 5 - 9\left(\frac{5-x}{9}\right) = 5 - (5-x) = x$$

The check works out so we know we did the work correctly and have inverse.

---

2. Given $g(x) = \frac{1}{2}x + 7$ find $g^{-1}(x)$.

Hint

Just follow the process outlines in the notes and you'll be set to do this problem!

Step 1

For the first step we simply replace the function with a y.

$$y = \frac{1}{2}x + 7$$

Step 2

Next, replace all the x's with y's and all the original y's with x's.

$$x = \frac{1}{2}y + 7$$

Step 3

Solve the equation from Step 2 for y.

$$\begin{aligned} x &= \frac{1}{2}y + 7 \\ 2x &= y + 14 \\ 2x - 14 &= y \end{aligned}$$

Step 4

Replace y with $g^{-1}(x)$.

$$g^{-1}(x) = 2x - 14$$

Step 5

Finally, do a quick check by computing one or both of $\left(g \circ g^{-1}\right)(x)$ and $\left(g^{-1} \circ g\right)(x)$ and verify that each is x. In general, we usually just check one of these and well do that here.

$$\left(g \circ g^{-1}\right)(x) = g\left[g^{-1}(x)\right] = g\left[2x - 14\right] = \frac{1}{2}(2x - 14) + 7 = x - 7 + 7 = x$$

The check works out so we know we did the work correctly and have inverse.

3. Given $f(x) = (x-2)^3 + 1$ find $f^{-1}(x)$.

Hint

Just follow the process outlines in the notes and you'll be set to do this problem!

Step 1

For the first step we simply replace the function with a y.

$$y = (x-2)^3 + 1$$

Step 2

Next, replace all the x's with y's and all the original y's with x's.

$$x = (y-2)^3 + 1$$

Step 3

Solve the equation from Step 2 for y.

$$\begin{aligned} x &= (y-2)^3 + 1 \\ x - 1 &= (y-2)^3 \\ (x-1)^{\frac{1}{3}} &= y - 2 \\ (x-1)^{\frac{1}{3}} + 2 &= y \end{aligned}$$

Step 4

Replace y with $f^{-1}(x)$.

$$f^{-1}(x) = (x-1)^{\frac{1}{3}} + 2$$

Step 5

Finally, do a quick check by computing one or both of $\left(f \circ f^{-1}\right)(x)$ and $\left(f^{-1} \circ f\right)(x)$ and verify that each is x. In general, we usually just check one of these and well do that here.

$$\begin{aligned} \left(f \circ f^{-1}\right)(x) = f\left[f^{-1}(x)\right] &= f\left[(x-1)^{\frac{1}{3}} + 2\right] \\ &= \left((x-1)^{\frac{1}{3}} + 2 - 2\right)^3 + 1 = \left((x-1)^{\frac{1}{3}}\right)^3 + 1 = x - 1 + 1 = x \end{aligned}$$

The check works out so we know we did the work correctly and have inverse.

4. Given $A(x) = \sqrt[5]{2x+11}$ find $A^{-1}(x)$.

Hint

Just follow the process outlines in the notes and you'll be set to do this problem!

Step 1

For the first step we simply replace the function with a y.

$$y = \sqrt[5]{2x+11}$$

Step 2

Next, replace all the x's with y's and all the original y's with x's.

$$x = \sqrt[5]{2y + 11}$$

Step 3

Solve the equation from Step 2 for y.

$$\begin{aligned} x^5 &= 2y + 11 \\ x^5 - 11 &= 2y \\ \frac{x^5 - 11}{2} &= y \end{aligned}$$

Step 4

Replace y with $A^{-1}(x)$.

$$A^{-1}(x) = \frac{x^5 - 11}{2}$$

Step 5

Finally, do a quick check by computing one or both of $\left(A \circ A^{-1}\right)(x)$ and $\left(A^{-1} \circ A\right)(x)$ and verify that each is x. In general, we usually just check one of these and well do that here.

$$\begin{aligned} \left(A \circ A^{-1}\right)(x) = A\left[A^{-1}(x)\right] = A\left[\frac{x^5 - 11}{2}\right] \\ = \sqrt[5]{2\left(\frac{x^5 - 11}{2}\right) + 11} = \sqrt[5]{x^5 - 11 + 11} = \sqrt[5]{x^5} = x \end{aligned}$$

The check works out so we know we did the work correctly and have inverse.

5. Given $f\left(x\right)=\dfrac{4x}{5-x}$ find $f^{-1}\left(x\right)$.

Hint

Just follow the process outlines in the notes and you'll be set to do this problem!

Step 1

For the first step we simply replace the function with a y.

$$y=\frac{4x}{5-x}$$

Step 2

Next, replace all the x's with y's and all the original y's with x's.

$$x=\frac{4y}{5-y}$$

Step 3

Solve the equation from Step 2 for y.

$$\begin{aligned}x&=\frac{4y}{5-y}\\ x\left(5-y\right)&=4y\\ 5x-xy&=4y\\ 5x&=4y+xy\\ 5x&=\left(4+x\right)y\\ \frac{5x}{4+x}&=y\end{aligned}$$

Step 4

Replace y with $f^{-1}\left(x\right)$.

$$f^{-1}\left(x\right)=\frac{5x}{4+x}$$

Step 5

Finally, do a quick check by computing one or both of $\left(f \circ f^{-1}\right)(x)$ and $\left(f^{-1} \circ f\right)(x)$ and verify that each is x. In general, we usually just check one of these and well do that here.

$$\left(f \circ f^{-1}\right)(x) = f\left[f^{-1}(x)\right] = f\left[\frac{5x}{4+x}\right]$$
$$= \frac{4\left(\frac{5x}{4+x}\right)}{5-\frac{5x}{4+x}}\,\frac{4+x}{4+x} = \frac{20x}{5(4+x)-5x} = \frac{20x}{20} = x$$

The check works out so we know we did the work correctly and have inverse.

6. Given $h(x) = \dfrac{1+2x}{7+x}$ find $h^{-1}(x)$.

Hint

Just follow the process outlines in the notes and you'll be set to do this problem!

Step 1

For the first step we simply replace the function with a y.

$$y = \frac{1+2x}{7+x}$$

Step 2

Next, replace all the x's with y's and all the original y's with x's.

$$x = \frac{1+2y}{7+y}$$

Step 3

Solve the equation from Step 2 for y.

$$\begin{aligned} x &= \frac{1+2y}{7+y} \\ x(7+y) &= 1+2y \\ 7x+xy &= 1+2y \\ 7x-1 &= 2y-xy \\ 7x-1 &= (2-x)\,y \\ \frac{7x-1}{2-x} &= y \end{aligned}$$

Step 4

Replace y with $h^{-1}(x)$.

$$h^{-1}(x) = \frac{7x-1}{2-x}$$

Step 5

Finally, do a quick check by computing one or both of $\left(h \circ h^{-1}\right)(x)$ and $\left(h^{-1} \circ h\right)(x)$ and verify that each is x. In general, we usually just check one of these and we'll do that here.

$$\begin{aligned} \left(h \circ h^{-1}\right)(x) = h\left[h^{-1}(x)\right] &= h\left[\frac{7x-1}{2-x}\right] \\ &= \frac{1+2\left(\frac{7x-1}{2-x}\right)}{7+\frac{7x-1}{2-x}}\,\frac{2-x}{2-x} = \frac{2-x+2(7x-1)}{7(2-x)+7x-1} = \frac{13x}{13} = x \end{aligned}$$

The check works out so we know we did the work correctly and have inverse.

4 Common Graphs

In this chapter we will continue the process of investigating graphing that we started in the last chapter. In the last chapter we discussed how to graph functions in general, lines, circles and piecewise functions. In this chapter we will take a look at parabolas, ellipses, hyperbolas, rational functions as well as a few other functions that will arise occasionally. In addition we will take a look at various transformations that we can make to functions and how these transformations impact the graph of the function. We will also introduce the concept of symmetry of a the graph of a function and how to determine if the function has symmetry without graphing the function.

The following sections are the practice problems, with solutions, for this material.

4.1 Lines, Circles and Piecewise Functions

We looked at these topics in the previous chapter. Problems for these topics can be found in the following sections.

Here are the appropriate sections to see for these.

Lines : Graphing and Functions - Lines

Circles : Graphing and Functions - Circles

Piecewise Functions : Graphing and Functions - Graphing Functions

4.2 Parabolas

1. Sketch the graph of the following parabola. The graph should contain the vertex, the y-intercept, x-intercepts (if any) and at least one point on either side of the vertex.

$$f(x) = (x+4)^2 - 3$$

Step 1

Let's find the vertex first. Because the parabola is in the form $f(x) = a(x-h)^2 + k$ we know that the vertex is just the point (h,k). Therefore, we can compare our equation to this form and see that the vertex is : $(-4,-3)$.

Be careful with minus signs here! For the h the term the general form is $(x-h)^2$ and so we need to write ours as $(x+4)^2 = (x-(-4))^2$. This in turn means we must have $h=-4$. Likewise, for the k in the general form it is "+k" and so to match our equation we need $k=-3$.

Also note that $a = 1 > 0$ for this parabola and so the parabola will open upwards.

Step 2

The y-intercept is just the point $(0, f(0))$. A quick function evaluation gives us that $f(0) = 13$ and so for our equation the y-intercept is $(0,13)$.

Step 3

For the x-intercepts we just need to solve the equation $f(x) = 0$. So, let's solve that for our equation.

$$\begin{aligned}(x+4)^2 - 3 &= 0\\ (x+4)^2 &= 3\\ x+4 &= \pm\sqrt{3}\\ x &= -4 \pm \sqrt{3}\end{aligned}$$

The two x-intercepts for this parabola are then : $\left(-4-\sqrt{3},0\right) = (-5.7321,0)$ and $\left(-4+\sqrt{3},0\right) = (-2.2679,0)$. Do not get excited about the "messy" values for the intercept coordinates. There is nothing wrong with them, they are just decimals rather than the integers we are used to dealing with.

Step 4

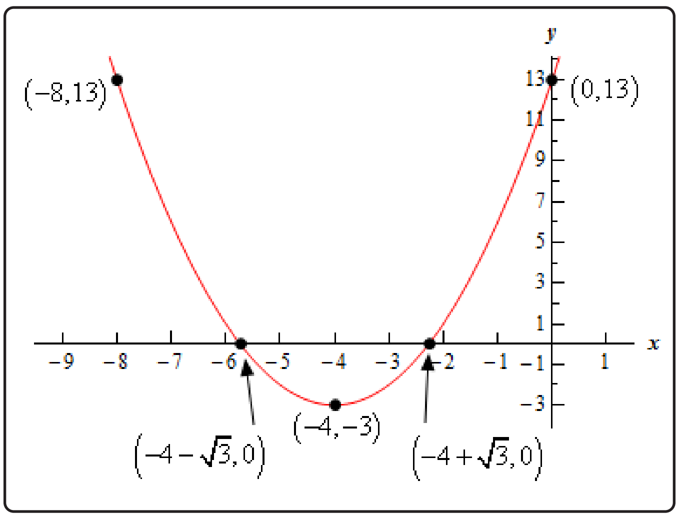
Because we had two x-intercepts for this parabola we already have at least one point on either side of the vertex and so we don't really need to find any more points for our graph.

However, just for the practice let's find the point corresponding to the y-intercept on the other side the vertex.

The y-intercept is a distance of 4 to the right of the vertex and so there will be a corresponding point at the same y value to the left and it will be a distance of 4 to the left of the vertex. Therefore, the point to the left of the vertex corresponding to the y-intercept is $(-8, 13)$.

Step 5

Here is a sketch of the parabola including all the points we found above.



2. Sketch the graph of the following parabola. The graph should contain the vertex, the y-intercept, x-intercepts (if any) and at least one point on either side of the vertex.

$$f(x) = 5(x-1)^2 - 20$$

Step 1

Let's find the vertex first. Because the parabola is in the form $f(x) = a(x-h)^2 + k$ we know that the vertex is just the point (h, k). Therefore, we can compare our equation to this form and see that the vertex is : $(1, -20)$.

Be careful with minus signs here! For the h the term the general form is $(x-h)^2$ and so to match our equation we must have $h = 1$. Likewise, for the k in the general form it is "+k" and so to match our equation we need $k = -20$.

Also note that $a = 5 > 0$ for this parabola and so the parabola will open upwards.

Step 2

The y-intercept is just the point $(0, f(0))$. A quick function evaluation gives us that $f(0) = -15$ and so for our equation the y-intercept is $(0, -15)$.

Step 3

For the x-intercepts we just need to solve the equation $f(x) = 0$. So, let's solve that for our equation.

$$\begin{aligned} 5(x-1)^2 - 20 &= 0 \\ 5(x-1)^2 &= 20 \\ (x-1)^2 &= 4 \\ x - 1 &= \pm 2 \\ x &= 1 \pm 2 = -1, 3 \end{aligned}$$

The two x-intercepts for this parabola are then : $(-1, 0)$ and $(3, 0)$.

Step 4

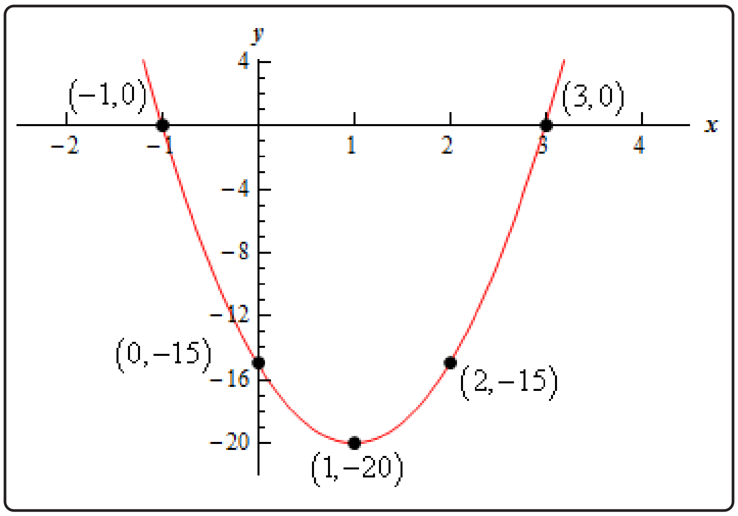
Because we had two x-intercepts for this parabola we already have at least one point on either side of the vertex and so we don't really need to find any more points for our graph.

However, just for the practice let's find the point corresponding to the y-intercept on the other side the vertex.

The y-intercept is a distance of 1 to the left of the vertex and so there will be a corresponding point at the same y value to the right and it will be a distance of 1 to the right of the vertex. Therefore, the point to the right of the vertex corresponding to the y-intercept is $(2, -15)$.

Step 5

Here is a sketch of the parabola including all the points we found above.



3. Sketch the graph of the following parabola. The graph should contain the vertex, the y-intercept, x-intercepts (if any) and at least one point on either side of the vertex.

$$f(x) = 3x^2 + 7$$

Step 1

Let's find the vertex first. In this case we can consider the equation to be in the form $f(x) = a(x-h)^2 + k$ or we can use the form $f(x) = ax^2 + bx + c$. Either form will work and which you find to be the easiest will probably depend on you. We'll use the first form so we can get another example of that form.

To make our equation up to the first form let's do a little rewrite on our equation. Let's write it as,

$$f(x) = 3x^2 + 7 = 3(x-0)^2 + 7$$

Note that we haven't changed the equation! All we've done is use $x = x - 0$ to get the forms to match up.

After doing this we can see that the vertex is : $(0, 7)$.

Also note that $a = 3 > 0$ for this parabola and so the parabola will open upwards.

Step 2

In this case we can see that the vertex above is on the y-axis (the x coordinate is zero!) and so it is also the y-intercept for the parabola!

Step 3

For the x-intercepts we just need to solve the equation $f(x) = 0$. So, let's solve that for our equation.

$$\begin{aligned} 3x^2 + 7 &= 0 \\ 3x^2 &= -7 \\ x^2 &= -\frac{7}{3} \\ x &= \pm\sqrt{-\frac{7}{3}} = \pm\sqrt{\frac{7}{3}}\,i \end{aligned}$$

So, in this case the solutions to this equation are complex numbers and so we know that this parabola will have no x-intercepts.

Note that we did not really need to solve the equation above to see that there would be no x-intercepts for this problem. An alternate method would be to do the following analysis.

From the first step we found that the vertex was $(0, 7)$, which is above the x-axis, and we also noted that the parabola opened upwards. So, the parabola starts above the x-axis and opens upwards and we know that once a parabola starts opening in a given direction it won't turn around and start going in the opposite direction. Therefore, because there is no way for the parabola to go *down* to the x-axis, there is no way for there to be x-intercepts for this problem.

Step 4

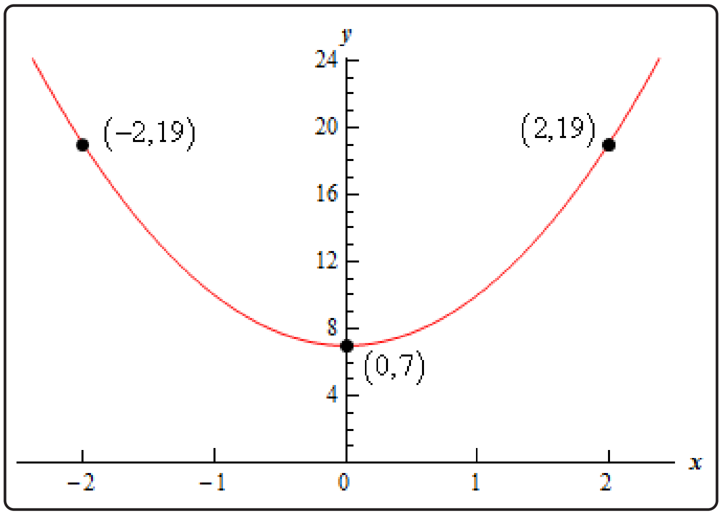
In this case we didn't have x-intercepts and the y-intercept also happens to be the vertex. So, at this point we have only have one point on the graph. To get points on either side of the vertex all we need to do is do a couple of quick function evaluations to find points on either side of the vertex.

We'll use the following two points.

$$(-2, f(-2)) = (-2, 19) \qquad\qquad (2, f(2)) = (2, 19)$$

Step 5

Here is a sketch of the parabola including all the points we found above.



4. Sketch the graph of the following parabola. The graph should contain the vertex, the y-intercept, x-intercepts (if any) and at least one point on either side of the vertex.

$$f(x) = x^2 + 12x + 11$$

Step 1

Let's find the vertex first. In this case the equation is in the form $f(x) = ax^2 + bx + c$. And so we know the vertex is the point $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$. The vertex is then,

$$\left(-\frac{12}{2(1)}, f\left(-\frac{12}{2(1)}\right)\right) = (-6, f(-6)) = (-6, -25)$$

Also note that $a = 1 > 0$ for this parabola and so the parabola will open upwards.

Step 2

The y-intercept is just the point $(0, f(0))$. A quick function evaluation gives us that $f(0) = 11$ and so for our equation the y-intercept is $(0, 11)$.

Step 3

For the x-intercepts we just need to solve the equation $f(x) = 0$. So, let's solve that for our equation.

$$\begin{aligned} x^2 + 12x + 11 &= 0 \\ (x+1)(x+11) &= 0 \qquad \rightarrow \qquad x = -1, \;\; x = -11 \end{aligned}$$

The two x-intercepts for this parabola are then : $(-1, 0)$ and $(-11, 0)$.

Step 4

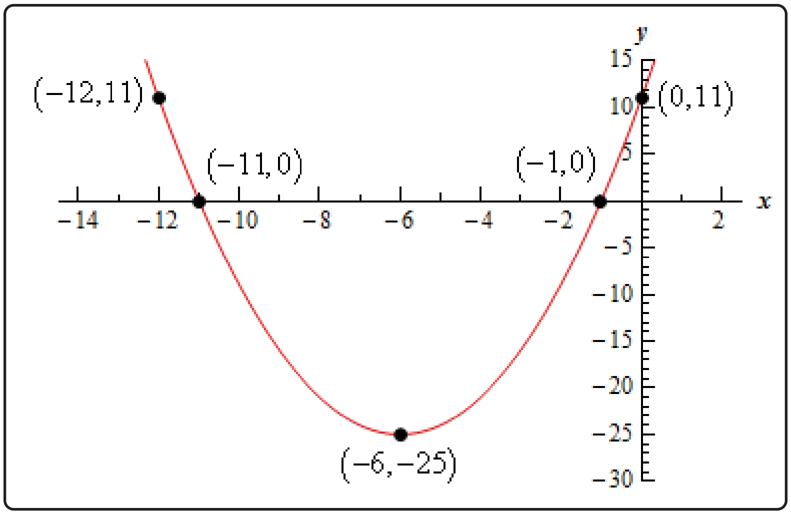
Because we had two x-intercepts for this parabola we already have at least one point on either side of the vertex and so we don't really need to find any more points for our graph.

However, just for the practice let's find the point corresponding to the y-intercept on the other side the vertex.

The y-intercept is a distance of 6 to the right of the vertex and so there will be a corresponding point at the same y value to the left and it will be a distance of 6 to the left of the vertex. Therefore, the point to the left of the vertex corresponding to the y-intercept is $(-12, 11)$.

Step 5

Here is a sketch of the parabola including all the points we found above.



5. Sketch the graph of the following parabola. The graph should contain the vertex, the y-intercept, x-intercepts (if any) and at least one point on either side of the vertex.

$$f(x) = 2x^2 - 12x + 26$$

Step 1

Let's find the vertex first. In this case the equation is in the form $f(x) = ax^2 + bx + c$. And so we know the vertex is the point $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$. The vertex is then,

$$\left(-\frac{-12}{2(2)}, f\left(-\frac{-12}{2(2)}\right)\right) = (3, f(3)) = (3, 8)$$

Also note that $a = 2 > 0$ for this parabola and so the parabola will open upwards.

Step 2

The y-intercept is just the point $(0, f(0))$. A quick function evaluation gives us that $f(0) = 26$ and so for our equation the y-intercept is $(0, 26)$.

Step 3

For the x-intercepts we just need to solve the equation $f(x) = 0$. So, let's solve that for our equation.

$$2x^2 - 12x + 26 = 0 \quad \rightarrow \quad x = \frac{12 \pm \sqrt{(-12)^2 - 4(2)(26)}}{2(2)} = \frac{12 \pm \sqrt{-64}}{4} = 3 \pm 2i$$

So, in this case the solutions to this equation are complex numbers and so we know that this parabola will have no x-intercepts.

Note that we did not really need to solve the equation above to see that there would be no x-intercepts for this problem. An alternate method would be to do the following analysis.

From the first step we found that the vertex was $(3, 8)$, which is above the x-axis, and we also noted that the parabola opened upwards. So, the parabola starts above the x-axis and opens upwards and we know that once a parabola starts opening in a given direction it won't turn around and start going in the opposite direction. Therefore, because there is no way for the parabola to go *down* to the x-axis, there is no way for there to be x-intercepts for this problem.

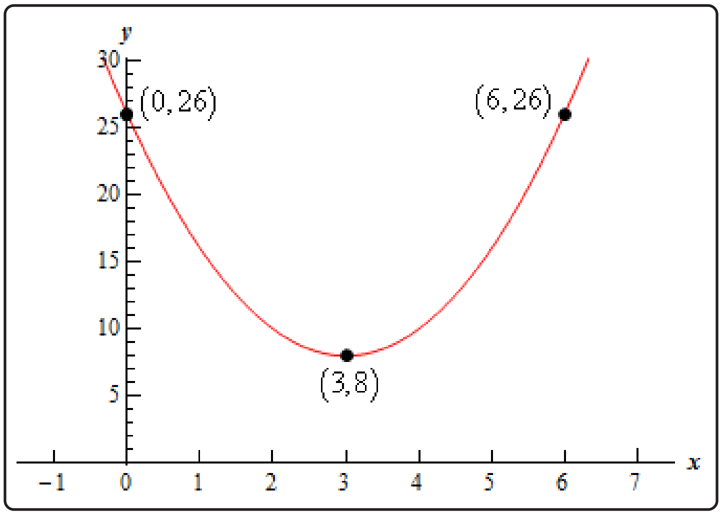
Step 4

In this case all we have are the vertex and the y-intercept (which is on the left side of the vertex). So, we'll need a point that is on the right side of the vertex and we can find the point on the left side of the vertex that corresponds to the y-intercept for this point.

The y-intercept is a distance of 3 to the left of the vertex and so there will be a corresponding point at the same y value to the right and it will be a distance of 3 to the right of the vertex. Therefore, the point to the right of the vertex corresponding to the *y*-intercept is $(6, 26)$.

Step 5

Here is a sketch of the parabola including all the points we found above.



6. Sketch the graph of the following parabola. The graph should contain the vertex, the y-intercept, x-intercepts (if any) and at least one point on either side of the vertex.

$$f(x) = 4x^2 - 4x + 1$$

Step 1

Let's find the vertex first. In this case the equation is in the form $f(x) = ax^2 + bx + c$. And so we know the vertex is the point $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$. The vertex is then,

$$\left(-\frac{-4}{2(4)}, f\left(-\frac{-4}{2(4)}\right)\right) = \left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right) = \left(\frac{1}{2}, 0\right)$$

Also note that $a = 4 > 0$ for this parabola and so the parabola will open upwards.

Step 2

The y-intercept is just the point $(0, f(0))$. A quick function evaluation gives us that $f(0) = 1$ and so for our equation the y-intercept is $(0, 1)$.

Step 3

For the x-intercepts we would normally solve the equation $f(x) = 0$. However, in this case we don't need to do that. From the first step we see that the vertex has a y-coordinate of zero and hence is also an x-intercept. Also, because it is the vertex this can be the only

x-intercept for this function.

Note that if we'd solved the equation we would have also arrived at this single x-intercept.

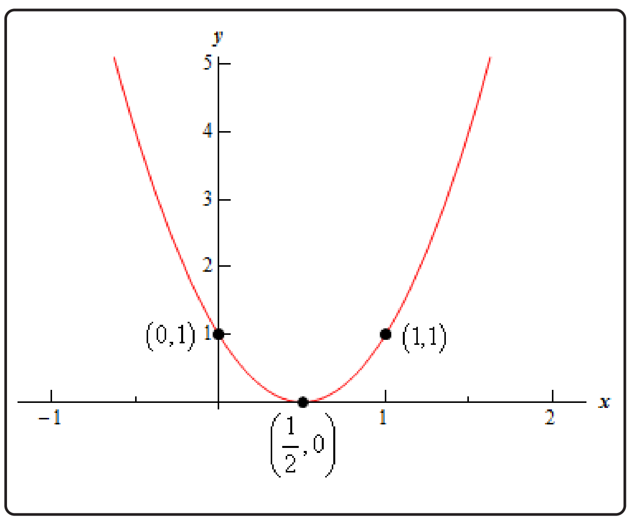
Step 4

In this case all we have are the vertex (which also happens to be the single x-intercept) and the y-intercept (which is on the right side of the vertex). So, we'll need a point that is on the left side of the vertex and we can find the point on the left side of the vertex that corresponds to the y-intercept for this point.

The y-intercept is a distance of 1 to the left of the vertex and so there will be a corresponding point at the same y value to the right and it will be a distance of 1 to the right of the vertex. Therefore, the point to the right of the vertex corresponding to the y-intercept is $(1,1)$.

Step 5

Here is a sketch of the parabola including all the points we found above.



7. Sketch the graph of the following parabola. The graph should contain the vertex, the y-intercept, x-intercepts (if any) and at least one point on either side of the vertex.

$$f(x) = -3x^2 + 6x + 3$$

Step 1

Let's find the vertex first. In this case the equation is in the form $f(x) = ax^2 + bx + c$. And so we know the vertex is the point $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$. The vertex is then,

$$\left(-\frac{6}{2(-3)}, f\left(-\frac{6}{2(-3)}\right)\right) = (1, f(1)) = (1, 6)$$

Also note that $a = -3 < 0$ for this parabola and so the parabola will open downwards.

Step 2

The y-intercept is just the point $(0, f(0))$. A quick function evaluation gives us that $f(0) = 3$ and so for our equation the y-intercept is $(0, 3)$.

Step 3

For the x-intercepts we just need to solve the equation $f(x) = 0$. So, let's solve that for our equation.

$$-3x^2 + 6x + 3 = 0 \quad \rightarrow \quad x = \frac{-6 \pm \sqrt{6^2 - 4(-3)(3)}}{2(-3)} = \frac{-6 \pm \sqrt{72}}{-6} = \frac{-6 \pm 6\sqrt{2}}{-6} = 1 \pm \sqrt{2}$$

The two x-intercepts for this parabola are then : $\left(1 - \sqrt{2}, 0\right) = (-0.4142, 0)$ and $\left(1 + \sqrt{2}, 0\right) = (2.4142, 0)$.

Step 4

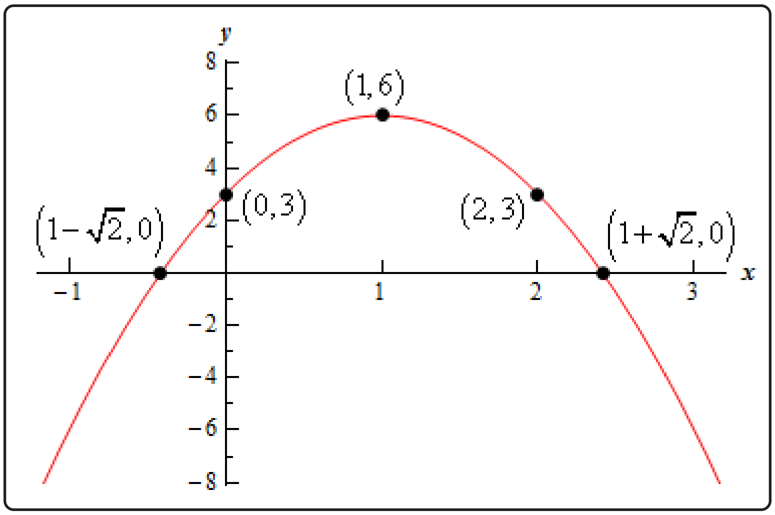
Because we had two x-intercepts for this parabola we already have at least one point on either side of the vertex and so we don't really need to find any more points for our graph.

However, just for the practice let's find the point corresponding to the y-intercept on the other side the vertex.

The y-intercept is a distance of 1 to the left of the vertex and so there will be a corresponding point at the same y value to the right and it will be a distance of 1 to the right of the vertex. Therefore, the point to the right of the vertex corresponding to the y-intercept is $(2, 3)$.

Step 5

Here is a sketch of the parabola including all the points we found above.



8. Convert the following equations into the form $y = a(x-h)^2 + k$.

$$f(x) = x^2 - 24x + 157$$

Step 1

We'll need to do the modified completing the square process described in the notes for this section.

The first step in this process is to make sure that we have a coefficient of one on the x^2, which we already have, so there is nothing we need to do in that regards for this problem.

Step 2

Next, we need to take one-half the coefficient of the x, square it and then add and subtract it onto the equation.

$$\left(\frac{-24}{2}\right)^2 = (-12)^2 = 144$$

$$f(x) = x^2 - 24x + 144 - 144 + 157$$

Step 3

Finally, all we need to do is factor the first three terms and combine the last two numbers to get,

$$\boxed{f(x) = (x-12)^2 + 13}$$

---

9. Convert the following equations into the form $y = a(x-h)^2 + k$.

$$f(x) = 6x^2 + 12x + 3$$

Step 1

We'll need to do the modified completing the square process described in the notes for this section.

The first step in this process is to make sure that we have a coefficient of one on the x^2. So, for this problem that means we need to factor a 6 out of the quadratic to get,

$$f(x) = 6\left(x^2 + 2x + \frac{1}{2}\right)$$

Be careful to not just cancel out a 6 from each term! We need to factor it out.

Step 2

Next, we need to take one-half the coefficient of the x, square it and then add and subtract it onto the equation.

$$\left(\frac{2}{2}\right)^2 = (1)^2 = 1$$

$$f(x) = 6\left(x^2 + 2x + 1 - 1 + \frac{1}{2}\right)$$

Make sure to do the adding/subtracting inside the parenthesis. If we did it outside of the parenthesis we would not be able to do the next step!

Step 3

Next, we need to factor the first three terms and combine the last two numbers to get,

$$f(x) = 6\left((x+1)^2 - \frac{1}{2}\right)$$

Step 4

Finally, all we need to do is multiply the 6 back through the parenthesis to get,

$$\boxed{f(x) = 6(x+1)^2 - 3}$$

10. Convert the following equations into the form $y = a(x-h)^2 + k$.

$$f(x) = -x^2 - 8x - 18$$

Step 1

We'll need to do the modified completing the square process described in the notes for this section.

The first step in this process is to make sure that we have a coefficient of one on the x^2. So, for this problem that means we need to factor a minus sign out of the quadratic to get,

$$f(x) = -\left(x^2 + 8x + 18\right)$$

Step 2

Next, we need to take one-half the coefficient of the x, square it and then add and subtract it onto the equation.

$$\left(\frac{8}{2}\right)^2 = (4)^2 = 16$$

$$f(x) = -\left(x^2 + 8x + 16 - 16 + 18\right)$$

Make sure to do the adding/subtracting inside the parenthesis. If we did it outside of the parenthesis we would not be able to do the next step!

Step 3

Next, we need to factor the first three terms and combine the last two numbers to get,

$$f(x) = -\left((x+4)^2 + 2\right)$$

Step 4

Finally, all we need to do is multiply the 6 back through the parenthesis to get,

$$\boxed{f\left(x\right) = -(x+4)^2 - 2}$$

4.3 Ellipses

1. Sketch the graph of the following ellipse.

$$\frac{(x+3)^2}{9}+\frac{(y-5)^2}{3}=1$$

Step 1

The first step here is to simply compare our equation to the standard form of the ellipse and identify all the important information. For reference purposes here is the standard form of the ellipse.

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

Comparing our equation to this we can see we have the following information.

$$h=-3 \qquad k=5 \qquad a=3 \qquad b=\sqrt{3}$$

Do, not worry about the square root in b. There is no reason to expect any of the numbers in the denominator to be perfect squares. They often aren't and so we need to be able to deal with them when they aren't. When we do get roots showing up it just means a couple of the points will be decimals instead of "nice" integers like we usually see with these kinds of problems.

Step 2

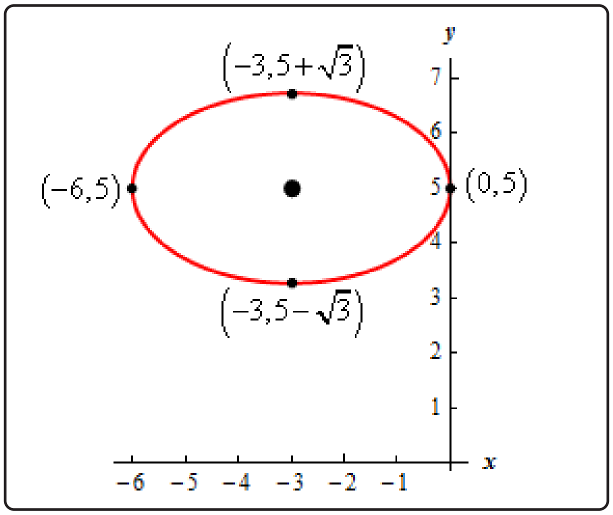
With the information we found in the first step we can see that the center of the ellipse is $(-3,5)$.

The right most, left most, top most and bottom most points are then,

Right Most Point : $(0,5)$
Left Most Point : $(-6,5)$
Top Most Point : $\left(-3,5+\sqrt{3}\right)=(-3,6.7321)$
Bottom Most Point : $\left(-3,5-\sqrt{3}\right)=(-3,3.2679)$

Step 3

Here is a sketch of the ellipse including all the points we found above.



2. Sketch the graph of the following ellipse.

$$x^2 + \frac{(y-1)^2}{4} = 1$$

Step 1

The first step here is to simply compare our equation to the standard form of the ellipse and identify all the important information. For reference purposes here is the standard form of the ellipse.

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

To help with the comparison let's rewrite our equation a little to make it look more like the standard form.

$$\frac{(x-0)^2}{1} + \frac{(y-1)^2}{4} = 1$$

We don't really need the "1" in the denominator nor do we need the "-0" in the numerator of the first term, but it might help to with the comparison process.

Comparing our equation to this we can see we have the following information.

$$h = 0 \qquad k = 1 \qquad a = 1 \qquad b = 2$$

Step 2

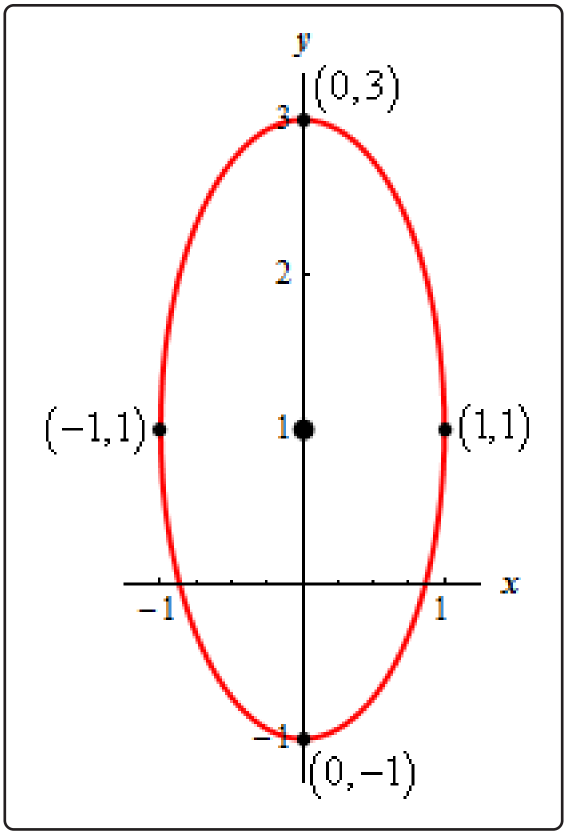
With the information we found in the first step we can see that the center of the ellipse is $(0,1)$.

The right most, left most, top most and bottom most points are then,

Right Most Point : $(1,1)$
Left Most Point : $(-1,1)$
Top Most Point : $(0,3)$
Bottom Most Point : $(0,-1)$

Step 3

Here is a sketch of the ellipse including all the points we found above.



3. Sketch the graph of the following ellipse.

$$4(x+2)^2 + \frac{(y+4)^2}{4} = 1$$

Step 1

The first step here is to simply compare our equation to the standard form of the ellipse and identify all the important information. For reference purposes here is the standard form of the ellipse.

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

To help with the comparison let's rewrite our equation a little to make it look more like the standard form.

$$\frac{(x+2)^2}{\frac{1}{4}} + \frac{(y+4)^2}{4} = 1$$

In order to properly identify a and b the numbers need to be in the denominator. So, we needed to move the 4 from the numerator of the first term into a ¼ in the denominator.

Comparing our equation to this we can see we have the following information.

$$h = -2 \qquad k = -4 \qquad a = \frac{1}{2} \qquad b = 2$$

Step 2

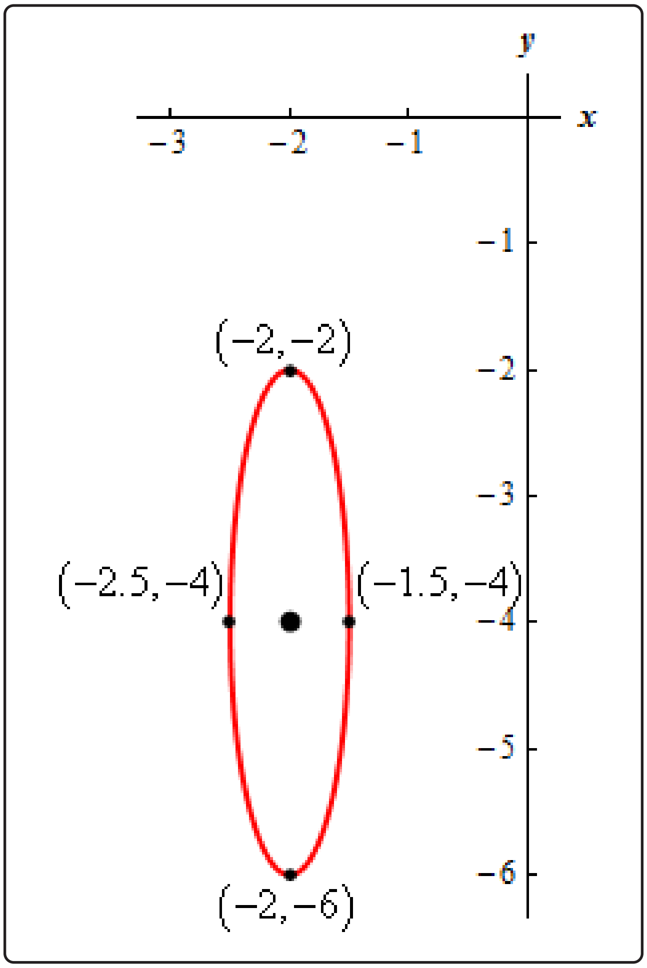
With the information we found in the first step we can see that the center of the ellipse is $(-2, -4)$.

The right most, left most, top most and bottom most points are then,

Right Most Point : $(-1.5, -4)$
Left Most Point : $(-2.5, -4)$
Top Most Point : $(-2, -2)$
Bottom Most Point : $(-2, -6)$

Step 3

Here is a sketch of the ellipse including all the points we found above.



4. Complete the square on the x and y portions of the equation and write the equation into the standard form of the equation of the ellipse.

$$x^2 + 8x + 3y^2 - 6y + 7 = 0$$

Step 1

The process here will be very similar to the process we used in the previous section to write equations of parabolas in standard form. We'll modify it slightly and also need to do an extra step or two but it is pretty similar to that process.

The first step is to make sure the coefficient of the x^2 and y^2 is a one. For the x^2 we already have a coefficient of one and so we don't need to do anything with that. The y^2 however has a coefficient of 3 and so we need to take care of that. What we will do is factor a 3 out of every term involving a y. Doing that gives,

$$x^2 + 8x + 3\left(y^2 - 2y\right) + 7 = 0$$

Note that unlike the parabola work we don't factor anything out of the constant term regardless of what we factor out of the x and/or y terms.

Step 2

Now let's get started on completing the square. First, we need one-half the coefficient of the x and y term, square each and the add/subtract those numbers in the appropriate places as follows,

$$\left(\frac{8}{2}\right)^2 = (4)^2 = 16 \qquad \left(\frac{-2}{2}\right)^2 = (-1)^2 = 1$$

$$x^2 + 8x + 16 - 16 + 3\left(y^2 - 2y + 1 - 1\right) + 7 = 0$$

Be careful with the y term! Make sure that when you add/subtract it you put it in the parenthesis!

Step 3

Next, we need to factor the x and y terms and add up all the constants.

$$\begin{aligned} (x+4)^2 - 16 + 3\left((y-1)^2 - 1\right) + 7 &= 0 \\ (x+4)^2 + 3(y-1)^2 + 7 - 16 - 3 &= 0 \\ (x+4)^2 + 3(y-1)^2 - 12 &= 0 \end{aligned}$$

When adding the constants up, make sure to multiply the 3 through the y terms before adding the constants up.

Step 4

To finish things off we'll first move the 12 to the other side of the equation.

$$(x+4)^2 + 3(y-1)^2 = 12$$

To get this into standard form we need a one on the right side of the equation. To get this all we need to do is divide everything by 12 and we'll do a little simplification work on the y term.

$$\frac{(x+4)^2}{12} + \frac{3(y-1)^2}{12} = 1 \quad \Rightarrow \quad \boxed{\frac{(x+4)^2}{12} + \frac{(y-1)^2}{4} = 1}$$

5. Complete the square on the x and y portions of the equation and write the equation into the standard form of the equation of the ellipse.

$$9x^2 + 126x + 4y^2 - 32y + 469 = 0$$

Step 1

The process here will be very similar to the process we used in the previous section to write equations of parabolas in standard form. We'll modify it slightly and also need to do an extra step or two but it is pretty similar to that process.

The first step is to make sure the coefficient of the x^2 and y^2 is a one. The x^2 has a coefficient of 9 and the y^2 has a coefficient of 4 and so we need to take care of each of those. What we will do is factor a 9 out of every term involving an x and factor a 4 out of every term involving a y. Doing that gives,

$$9\left(x^2 + 14x\right) + 4\left(y^2 - 8y\right) + 469 = 0$$

Note that unlike the parabola work we don't factor anything out of the constant term regardless of what we factor out of the x and/or y terms.

Step 2

Now let's get started on completing the square. First, we need one-half the coefficient of the x and y term, square each and the add/subtract those numbers in the appropriate places as follows,

$$\left(\frac{14}{2}\right)^2 = (7)^2 = 49 \qquad \left(\frac{-8}{2}\right)^2 = (-4)^2 = 16$$

$$9\left(x^2 + 14x + 49 - 49\right) + 4\left(y^2 - 8y + 16 - 16\right) + 469 = 0$$

Be careful you add/subtract these numbers and make sure you put them in the parenthesis!

Step 3

Next, we need to factor the x and y terms and add up all the constants.

$$\begin{aligned} 9\left((x+7)^2 - 49\right) + 4\left((y-4)^2 - 16\right) + 469 &= 0 \\ 9(x+7)^2 + 4(y-4)^2 + 469 - 441 - 64 &= 0 \\ 9(x+7)^2 + 4(y-4)^2 - 36 &= 0 \end{aligned}$$

When adding the constants up, make sure to multiply the 9 through the x terms and the 4 through the y terms before adding the constants up.

Step 4

To finish things off we'll first move the 36 to the other side of the equation.

$$9\left((x+7)^2\right)+4\left((y-4)^2\right)=36$$

To get this into standard form we need a one on the right side of the equation. To get this all we need to do is divide everything by 36 and we'll do a little simplification work on the terms.

$$\frac{9\left((x+7)^2\right)}{36}+\frac{4\left((y-4)^2\right)}{36}=1 \qquad \Rightarrow \qquad \boxed{\frac{(x+7)^2}{4}+\frac{(y-4)^2}{9}=1}$$

4.4 Hyperbolas

1. Sketch the graph of the following hyperbola.

$$\frac{y^2}{16} - \frac{(x-2)^2}{9} = 1$$

Step 1

The first step here is to simply compare our equation to the standard form of the hyperbola and identify all the important information. For reference purposes here is the standard form of the hyperbola that matches the one we have here.

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

Comparing our equation to this we can see we have the following information.

$$h = 2 \qquad k = 0 \qquad a = 3 \qquad b = 4$$

Because the y term is the positive term we know that this hyperbola will open up and down.

Step 2

With the information we found in the first step we can see that the center of the hyperbola is $(2, 0)$.

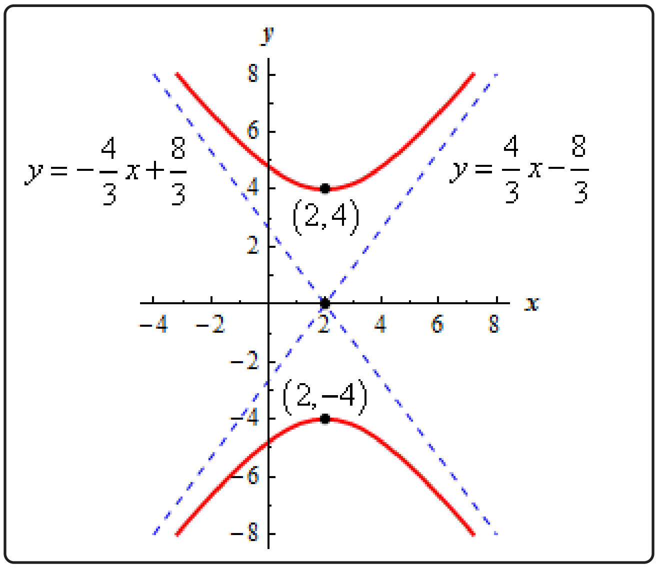
The vertices of hyperbola are : $(2, -4)$ and $(2, 4)$.

The equations of the two asymptotes are,

$$y = 0 + \frac{4}{3}(x-2) = \frac{4}{3}x - \frac{8}{3} \qquad y = 0 - \frac{4}{3}(x-2) = -\frac{4}{3}x + \frac{8}{3}$$

Step 3

Here is a sketch of the hyperbola including the points and asymptotes we found above.



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2. Sketch the graph of the following hyperbola.

$$\frac{(x+3)^2}{4}-\frac{(y-1)^2}{9}=1$$

Step 1

The first step here is to simply compare our equation to the standard form of the hyperbola and identify all the important information. For reference purposes here is the standard form of the hyperbola that matches the one we have here.

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

Comparing our equation to this we can see we have the following information.

$$h=-3 \qquad k=1 \qquad a=2 \qquad b=3$$

Because the x term is the positive term we know that this hyperbola will open right and left.

Step 2

With the information we found in the first step we can see that the center of the hyperbola is $(-3, 1)$.

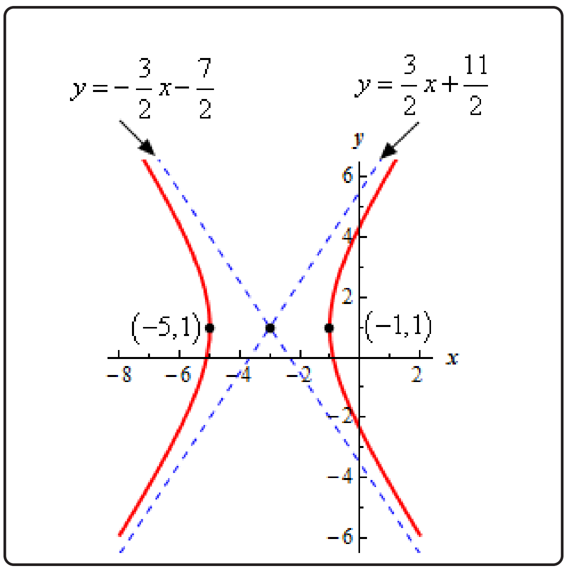
The vertices of hyperbola are : $(-5, 1)$ and $(-1, 1)$.

The equations of the two asymptotes are,

$$y = 1 + \frac{3}{2}(x+3) = \frac{3}{2}x + \frac{11}{2} \qquad y = 1 - \frac{3}{2}(x+3) = -\frac{3}{2}x - \frac{7}{2}$$

Step 3

Here is a sketch of the hyperbola including the points and asymptotes we found above.



3. Sketch the graph of the following hyperbola.

$$3(x-1)^2 - \frac{(y+1)^2}{2} = 1$$

Step 1

The first step here is to simply compare our equation to the standard form of the hyperbola and identify all the important information. For reference purposes here is the standard form of the hyperbola that matches the one we have here.

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

To help with the comparison let's rewrite our equation a little to make it look more like the standard form.

$$\frac{(x-1)^2}{\frac{1}{3}} - \frac{(y+1)^2}{2} = 1$$

In order to properly identify a and b the numbers need to be in the denominator. So, we needed to move the 3 from the numerator of the first term into a 1/3 in the denominator.

Comparing our equation to this we can see we have the following information.

$$h = 1 \qquad k = -1 \qquad a = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}} \qquad b = \sqrt{2}$$

Do not get excited about the square roots in the a and b. They will be present on occasion and so we need to be able to deal with them. Also, in order to make it slightly easier to deal with we did simplfy the a by taking the square root of both the numerator (which was really easy to compute) and the denominator.

Because the x term is the positive term we know that this hyperbola will open right and left.

Step 2

With the information we found in the first step we can see that the center of the hyperbola is $(1, -1)$.

The vertices of hyperbola are : $\left(1 - \frac{1}{\sqrt{3}}, -1\right) = (0.4226, -1)$ and $\left(1 + \frac{1}{\sqrt{3}}, -1\right) = (1.5774, -1)$.

The equations of the two asymptotes are,

$$y = -1 + \frac{\sqrt{2}}{1/\sqrt{3}}(x-1) = \sqrt{6}\,x - \sqrt{6} - 1 \qquad y = -1 - \frac{\sqrt{2}}{1/\sqrt{3}}(x-1) = -\sqrt{6}\,x + \sqrt{6} - 1$$

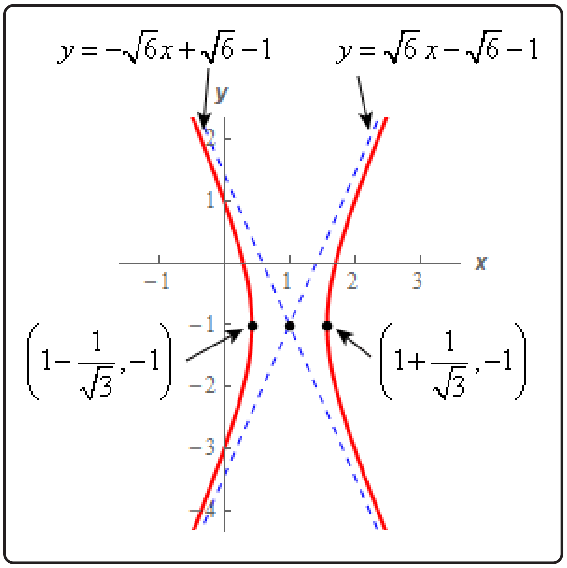
The asymptotes in this case are somewhat messier than we are used to dealing with because of the square roots but they work the same way all lines do. The only difference is the numbers are simply messier.

Also note that we simplified the slopes as follows,

$$\frac{\sqrt{2}}{1/\sqrt{3}} = \sqrt{2}\,\frac{\sqrt{3}}{1} = \sqrt{2\,(3)} = \sqrt{6}$$

Step 3

Here is a sketch of the hyperbola including the points and asymptotes we found above.



4. Complete the square on the x and y portions of the equation and write the equation into the standard form of the equation of the hyperbola.

$$4x^2 - 32x - y^2 - 4y + 24 = 0$$

Step 1

The process here will be is identical to the process we used in the previous section to write equations of ellipses in standard form.

The first step is to make sure the coefficient of the x^2 and y^2 is a one. The x^2 has a coefficient of 4 and the y^2 has a coefficient of -1. What we will do is factor a 4 out of every term involving an x and a -1 out of ever term involving a y. Doing that gives,

$$4\left(x^2 - 8x\right) - \left(y^2 + 4y\right) + 24 = 0$$

Be careful with these kinds of problems and don't forget that even a coefficient of -1 needs to be taken care of!

Step 2

Now let's get started on completing the square. First, we need one-half the coefficient of the x and y term, square each and the add/subtract those numbers in the appropriate places as follows,

$$\left(\frac{-8}{2}\right)^2 = (-4)^2 = 16 \qquad \left(\frac{4}{2}\right)^2 = (2)^2 = 4$$

$$4\left(x^2 - 8x + 16 - 16\right) - \left(y^2 + 4y + 4 - 4\right) + 24 = 0$$

Be careful you add/subtract these numbers and make sure you put them in the parenthesis!

Step 3

Next, we need to factor the x and y terms and add up all the constants.

$$\begin{aligned} 4\left((x-4)^2 - 16\right) - \left((y+2)^2 - 4\right) + 24 &= 0 \\ 4(x-4)^2 - (y+2)^2 + 24 - 64 + 4 &= 0 \\ 4(x-4)^2 - (y+2)^2 - 36 &= 0 \end{aligned}$$

When adding the constants up, make sure to multiply the 4 through the x terms and the -1 through the y terms before adding the constants up.

Step 4

To finish things off we'll first move the 36 to the other side of the equation.

$$4(x-4)^2 - (y+2)^2 = 36$$

To get this into standard form we need a one on the right side of the equation. To get this all we need to do is divide everything by 36 and we'll do a little simplification work on the x term.

$$\frac{4(x-4)^2}{36} - \frac{(y+2)^2}{36} = 1 \qquad \Rightarrow \qquad \boxed{\frac{(x-4)^2}{9} - \frac{(y+2)^2}{36} = 1}$$

5. Complete the square on the x and y portions of the equation and write the equation into the standard form of the equation of the hyperbola.

$$25y^2 + 250y - 16x^2 - 32x + 209 = 0$$

Step 1

The process here will be is identical to the process we used in the previous section to write equations of ellipses in standard form.

The first step is to make sure the coefficient of the x^2 and y^2 is a one. The x^2 has a coefficient of -16 and the y^2 has a coefficient of 25. What we will do is factor a -16 out of every term involving an x and a 25 out of ever term involving a y. Doing that gives,

$$25\left(y^2 + 10y\right) - 16\left(x^2 + 2x\right) + 209 = 0$$

Step 2

Now let's get started on completing the square. First, we need one-half the coefficient of the x and y term, square each and the add/subtract those numbers in the appropriate places as follows,

$$\left(\frac{10}{2}\right)^2 = (5)^2 = 25 \qquad \left(\frac{2}{2}\right)^2 = (1)^2 = 1$$

$$25\left(y^2 + 10y + 25 - 25\right) - 16\left(x^2 + 2x + 1 - 1\right) + 209 = 0$$

Be careful you add/subtract these numbers and make sure you put them in the parenthesis!

Step 3

Next, we need to factor the x and y terms and add up all the constants.

$$\begin{aligned} 25\left((y+5)^2 - 25\right) - 16\left((x+1)^2 - 1\right) + 209 &= 0 \\ 25(y+5)^2 - 16(x+1)^2 + 209 - 625 + 16 &= 0 \\ 25(y+5)^2 - 16(x+1)^2 - 400 &= 0 \end{aligned}$$

When adding the constants up, make sure to multiply the -16 through the x terms and the 25 through the y terms before adding the constants up.

Step 4

To finish things off we'll first move the 400 to the other side of the equation.

$$25(y+5)^2 - 16(x+1)^2 = 400$$

To get this into standard form we need a one on the right side of the equation. To get this all we need to do is divide everything by 400 and we'll do a little simplification work on the terms.

$$\frac{25(y+5)^2}{400} - \frac{16(x+1)^2}{400} = 1 \qquad \Rightarrow \qquad \boxed{\frac{(y+5)^2}{16} - \frac{(x+1)^2}{25} = 1}$$

4.5 Miscellaneous Functions

The sole purpose of this section was to get you familiar with the basic shape of some miscellaneous functions for the next section. As such there are no problems for this section. You will see quite a few problems utilizing these functions in the Transformations section.

4.6 Transformations

1. Use transformations to sketch the graph of the following function.

$$f(x) = \sqrt{x} + 4$$

Step 1

Let's first identify the "base" function (*i.e.* the function we are transforming). In this case it looks like we are transforming $g(x) = \sqrt{x}$.

Step 2

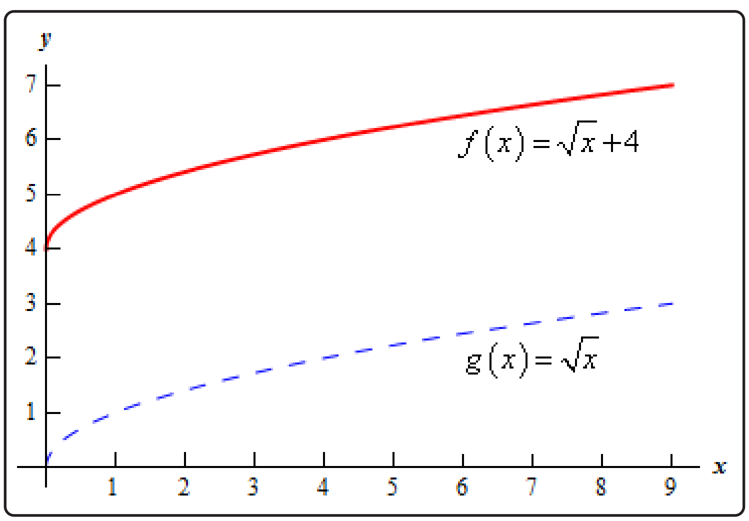
The function that have here looks like it can be written as,

$$f(x) = \sqrt{x} + 4 = g(x) + 4$$

Therefore, we can see that the graph of $f(x)$ is simply going to be the graph of $g(x)$ shifted up by 4.

Step 3

Here is a sketch of both the base function (blue dashed curve) and the function we were asked to graph (red solid curve).



2. Use transformations to sketch the graph of the following function.

$$f(x) = x^3 - 2$$

Step 1

Let's first identify the "base" function (*i.e.* the function we are transforming). In this case it looks like we are transforming $g(x) = x^3$.

Step 2

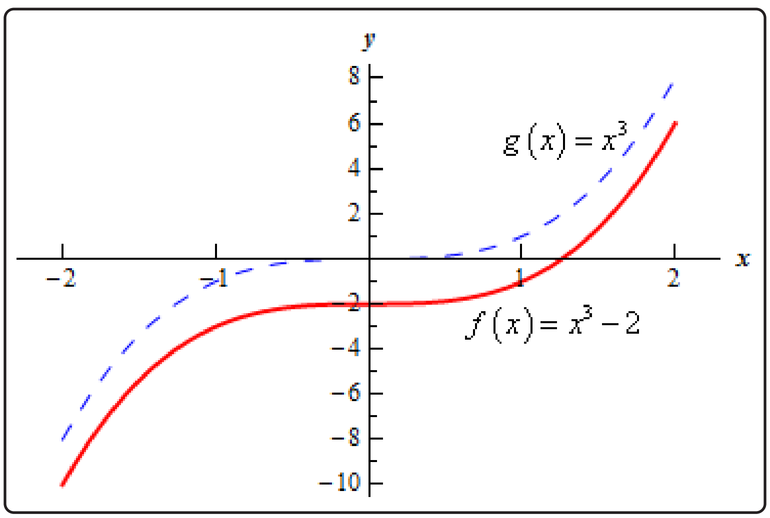
The function that have here looks like it can be written as,

$$f(x) = x^3 - 2 = g(x) - 2$$

Therefore, we can see that the graph of $f(x)$ is simply going to be the graph of $g(x)$ shifted down by 2.

Step 3

Here is a sketch of both the base function (blue dashed curve) and the function we were asked to graph (red solid curve).



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3. Use transformations to sketch the graph of the following function.

$$f(x) = |x + 2|$$

Step 1

Let's first identify the "base" function (*i.e.* the function we are transforming). In this case it looks like we are transforming $g(x) = |x|$.

Step 2

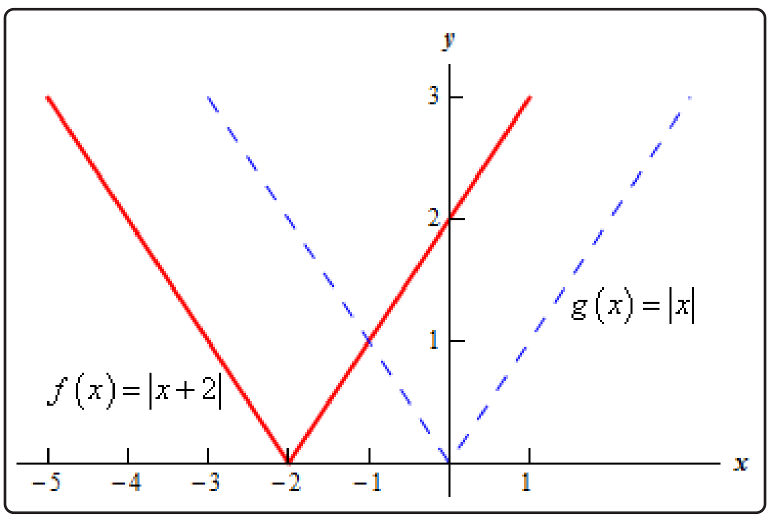
The function that have here looks like it can be written as,

$$f(x) = |x+2| = g(x+2)$$

Therefore, we can see that the graph of $f(x)$ is simply going to be the graph of $g(x)$ shifted left by 2.

Step 3

Here is a sketch of both the base function (blue dashed curve) and the function we were asked to graph (red solid curve).



4. Use transformations to sketch the graph of the following function.

$$f(x) = (x-5)^2$$

Step 1

Let's first identify the "base" function (*i.e.* the function we are transforming). In this case it looks like we are transforming $g(x) = x^2$.

Step 2

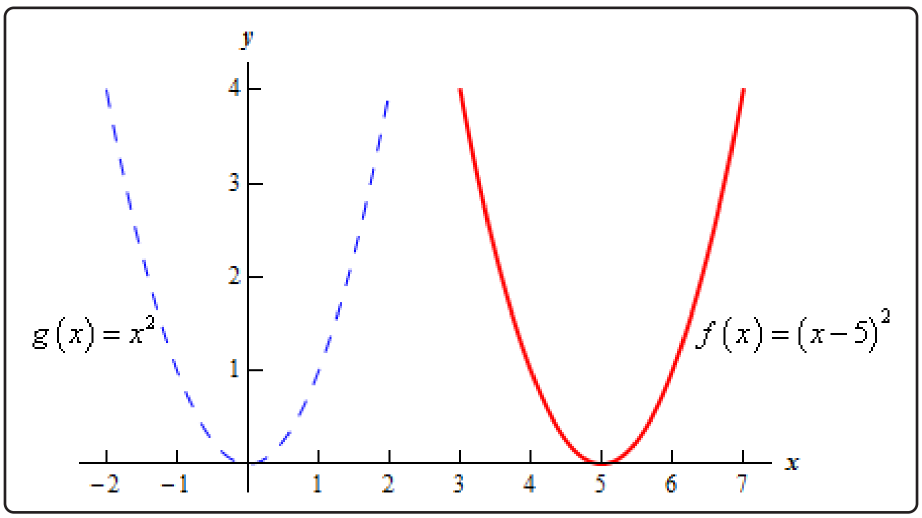
The function that have here looks like it can be written as,

$$f(x) = (x-5)^2 = g(x-5)$$

Therefore, we can see that the graph of $f(x)$ is simply going to be the graph of $g(x)$ shifted right by 5.

Step 3

Here is a sketch of both the base function (blue dashed curve) and the function we were asked to graph (red solid curve).



5. Use transformations to sketch the graph of the following function.

$$f(x) = -x^3$$

Step 1

Let's first identify the "base" function (*i.e.* the function we are transforming). In this case it looks like we are transforming $g(x) = x^3$.

Step 2

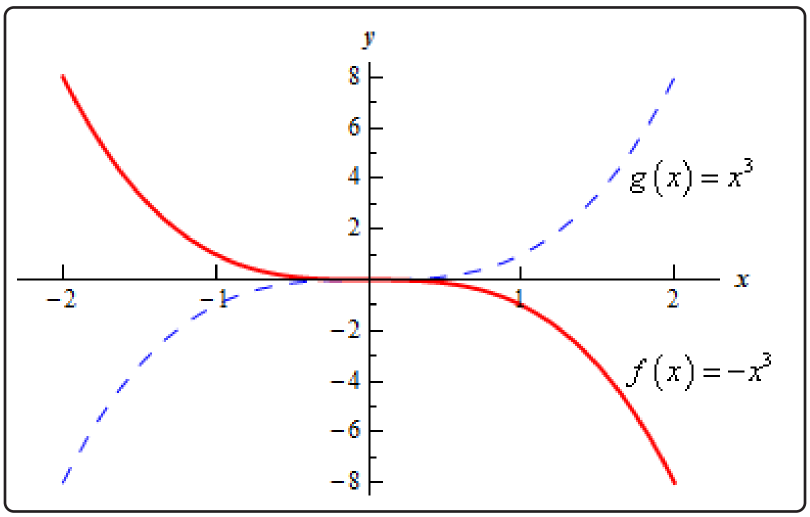
The function that have here looks like it can be written as,

$$f(x) = -x^3 = -g(x)$$

Therefore, we can see that the graph of $f(x)$ is simply going to be the graph of $g(x)$ reflected about the x-axis.

Step 3

Here is a sketch of both the base function (blue dashed curve) and the function we were asked to graph (red solid curve).



6. Use transformations to sketch the graph of the following function.

$$f(x) = \sqrt{x+4} - 3$$

Step 1

Let's first identify the "base" function (*i.e.* the function we are transforming). In this case it looks like we are transforming $g(x) = \sqrt{x}$.

Step 2

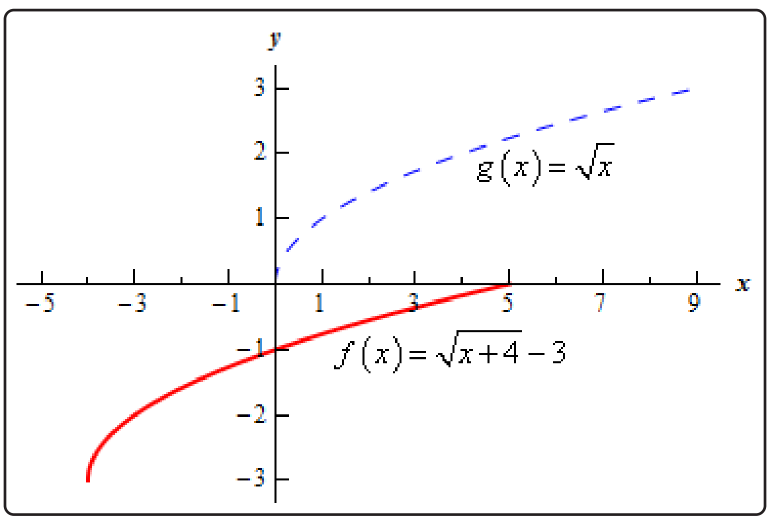
The function that have here looks like it can be written as,

$$f(x) = \sqrt{x+4} - 3 = g(x+4) - 3$$

Therefore, we can see that the graph of $f(x)$ is simply going to be the graph of $g(x)$ shifted left by 4 and down by 3.

Step 3

Here is a sketch of both the base function (blue dashed curve) and the function we were asked to graph (red solid curve).



7. Use transformations to sketch the graph of the following function.

$$f(x) = |x-7| + 2$$

Step 1

Let's first identify the "base" function (*i.e.* the function we are transforming). In this case it looks like we are transforming $g(x) = |x|$.

Step 2

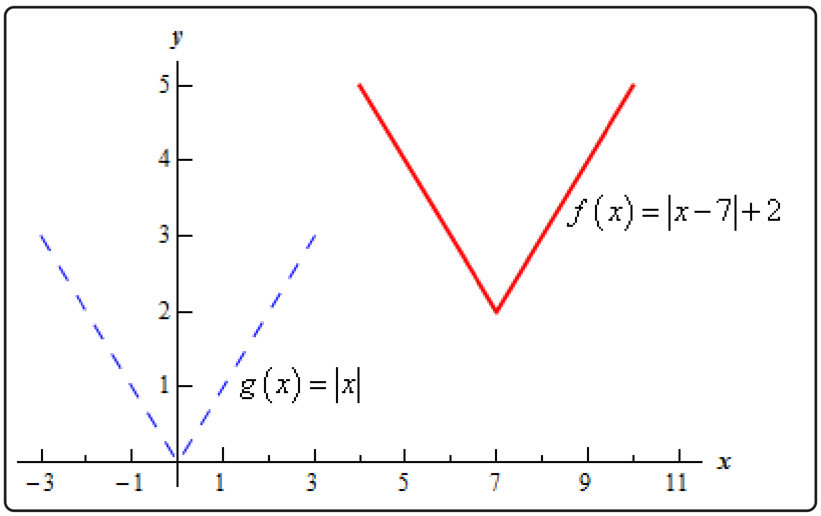
The function that have here looks like it can be written as,

$$f(x) = |x - 7| + 2 = g(x - 7) + 2$$

Therefore, we can see that the graph of $f(x)$ is simply going to be the graph of $g(x)$ shifted right by 7 and up by 2.

Step 3

Here is a sketch of both the base function (blue dashed curve) and the function we were asked to graph (red solid curve).



4.7 Symmetry

1. Determine the symmetry of each of the following equation.

$$x = 4y^6 - y^2$$

Step 1

Let's first check for symmetry about the x-axis. To do this we need to replace all the y's with -y.

$$x = 4(-y)^6 - (-y)^2 \qquad \rightarrow \qquad x = 4y^6 - y^2$$

The resulting equation is identical to the original equation and so is equivalent to the original equation. Therefore, the equation is **has symmetry about the x-axis**.

Step 2

Next, we'll check for symmetry about the y-axis. To do this we need to replace all the x's with $-x$.

$$-x = 4y^6 - y^2$$

The resulting equation is not equivalent to the original equation (*i.e.* it is not same nor is it the same equation except with opposite signs on every term). Therefore, the equation is **does not have symmetry about the y-axis**.

Step 3

Finally, a check for symmetry about the origin. For this check we need to replace all the y's with $-y$ and to replace all the x's with $-x$.

$$-x = 4(-y)^6 - (-y)^2 \qquad \rightarrow \qquad -x = 4y^6 - y^2$$

The resulting equation is not equivalent to the original equation (*i.e.* it is not same nor is it the same equation except with opposite signs on every term). Therefore, the equation **does not have symmetry about the origin**.

2. Determine the symmetry of each of the following equation.

$$\frac{y^2}{4} = 1 + \frac{x^2}{9}$$

Step 1

Let's first check for symmetry about the x-axis. To do this we need to replace all the y's with $-y$.

$$\frac{(-y)^2}{4} = 1 + \frac{x^2}{9} \qquad \to \qquad \frac{y^2}{4} = 1 + \frac{x^2}{9}$$

The resulting equation is identical to the original equation and so is equivalent to the original equation. Therefore, the equation is **has symmetry about the x-axis**.

Step 2

Next, we'll check for symmetry about the y-axis. To do this we need to replace all the x's with $-x$.

$$\frac{y^2}{4} = 1 + \frac{(-x)^2}{9} \qquad \to \qquad \frac{y^2}{4} = 1 + \frac{x^2}{9}$$

The resulting equation is identical to the original equation and so is equivalent to the original equation. Therefore, the equation is **has symmetry about the y-axis**.

Step 3

Finally, a check for symmetry about the origin. For this check we need to replace all the y's with $-y$ and to replace all the x's with $-x$.

$$\frac{(-y)^2}{4} = 1 + \frac{(-x)^2}{9} \qquad \to \qquad \frac{y^2}{4} = 1 + \frac{x^2}{9}$$

The resulting equation is identical to the original equation and so is equivalent to the original equation. Therefore, the equation is **has symmetry about the origin**.

3. Determine the symmetry of each of the following equation.

$$x^2 = 7y - x^3 + 2$$

Step 1

Let's first check for symmetry about the x-axis. To do this we need to replace all the y's with $-y$.

$$x^2 = 7(-y) - x^3 + 2 \qquad \to \qquad x^2 = -7y - x^3 + 2$$

The resulting equation is not equivalent to the original equation (*i.e.* it is not same nor is it the same equation except with opposite signs on every term). Therefore, the equation is **does not have symmetry about the x-axis**.

Step 2

Next, we'll check for symmetry about the y-axis. To do this we need to replace all the x's with $-x$.

$$(-x)^2 = 7y - (-x)^3 + 2 \qquad \to \qquad x^2 = 7y + x^3 + 2$$

The resulting equation is not equivalent to the original equation (*i.e.* it is not same nor is it the same equation except with opposite signs on every term). Therefore, the equation is **does not have symmetry about the y-axis**.

Step 3

Finally, a check for symmetry about the origin. For this check we need to replace all the y's with $-y$ and to replace all the x's with $-x$.

$$(-x)^2 = 7(-y) - (-x)^3 + 2 \qquad \to \qquad x^2 = -7y + x^3 + 2$$

The resulting equation is not equivalent to the original equation (*i.e.* it is not same nor is it the same equation except with opposite signs on every term). Therefore, the equation **does not have symmetry about the origin**.

4. Determine the symmetry of each of the following equation.

$$y = 4x^2 + x^6 - x^8$$

Step 1

Let's first check for symmetry about the x-axis. To do this we need to replace all the y's with $-y$.

$$-y = 4x^2 + x^6 - x^8$$

The resulting equation is not equivalent to the original equation (*i.e.* it is not same nor is it the same equation except with opposite signs on every term). Therefore, the equation is **does not have symmetry about the x-axis**.

Step 2

Next, we'll check for symmetry about the y-axis. To do this we need to replace all the x's with $-x$.

$$y = 4(-x)^2 + (-x)^6 - (-x)^8 \qquad \to \qquad y = 4x^2 + x^6 - x^8$$

The resulting equation is identical to the original equation and so is equivalent to the original equation. Therefore, the equation is **has symmetry about the y-axis**.

Step 3

Finally, a check for symmetry about the origin. For this check we need to replace all the y's with $-y$ and to replace all the x's with $-x$.

$$-y = 4(-x)^2 + (-x)^6 - (-x)^8 \qquad \to \qquad -y = 4x^2 + x^6 - x^8$$

The resulting equation is not equivalent to the original equation (*i.e.* it is not same nor is it the same equation except with opposite signs on every term). Therefore, the equation **does not have symmetry about the origin**.

5. Determine the symmetry of each of the following equation.

$$y = 7x + 4x^5$$

Step 1

Let's first check for symmetry about the x-axis. To do this we need to replace all the y's with $-y$.

$$-y = 7x + 4x^5$$

The resulting equation is not equivalent to the original equation (*i.e.* it is not same nor is it the same equation except with opposite signs on every term). Therefore, the equation is **does not have symmetry about the x-axis**.

Step 2

Next, we'll check for symmetry about the y-axis. To do this we need to replace all the x's with $-x$.

$$y = 7(-x) + 4(-x)^5 \qquad \to \qquad y = -7x - 4x^5$$

The resulting equation is not equivalent to the original equation (*i.e.* it is not same nor is it the same equation except with opposite signs on every term). Therefore, the equation is **does not have symmetry about the y-axis**.

Step 3

Finally, a check for symmetry about the origin. For this check we need to replace all the y's with $-y$ and to replace all the x's with $-x$.

$$-y = 7(-x) + 4(-x)^5 \qquad \to \qquad -y = -7x - 4x^5$$

The resulting equation is the same as the original equation except all the signs are the opposite and so is equivalent to the original equation. Therefore, the equation **has symmetry about the origin**.

4.8 Rational Functions

1. Sketch the graph of the following function. Clearly identify all intercepts and asymptotes.

$$f(x) = \frac{-4}{x-2}$$

Step 1

Let's first find the intercepts for this function.

The y-intercept is the point $(0, f(0)) = (0, 2)$.

For the x-intercepts we set the numerator equal to zero and solve. However, in this case the numerator is a constant (-4 specifically) and so can't ever be zero. Therefore, this function will have no x-intercepts.

Step 2

We can find any vertical asymptotes be setting the denominator equal to zero and solving. Doing that for this function gives,

$$x - 2 = 0 \qquad \rightarrow \qquad x = 2$$

So, we'll have a vertical asymptote at $x = 2$.

Step 3

For this equation the largest exponent of x in the numerator is zero since the numerator is a constant. The largest exponent of x in the denominator is 1, which is larger than the largest exponent in the numerator, and so the x-axis will be the horizontal asymptote.

Step 4

From Step 2 we saw we only have one vertical asymptote and so we only have two regions to our graph : $x < 2$ and $x > 2$.

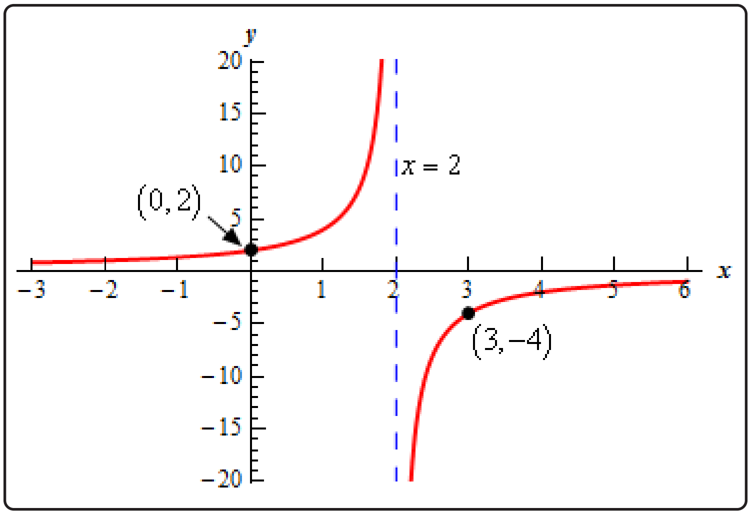
We'll need a point in each region to determine if it will be above or below the horizontal asymptote. Here are a couple of function evaluations for the points.

$$\begin{aligned} f(0) &= 2 & \rightarrow \qquad & (0, 2) \\ f(3) &= -4 & \rightarrow \qquad & (3, -4) \end{aligned}$$

Note that the first evaluation didn't really need to be done since it was just the y-intercept which we had already found in the first step. It was included here mostly for the sake of completeness.

Step 5

Here is a sketch of the function with the points found above. The vertical asymptote is indicated with a blue dashed line and recall that the horizontal asymptote is just the x-axis.



2. Sketch the graph of the following function. Clearly identify all intercepts and asymptotes.

$$f(x) = \frac{6-2x}{1-x}$$

Step 1

Let's first find the intercepts for this function.

The y-intercept is the point $(0, f(0)) = (0, 6)$.

For the x-intercepts we set the numerator equal to zero and solve. Doing that for this problem gives,

$$6 - 2x = 0 \qquad \rightarrow \qquad x = 3$$

So, the only x-intercept for this problem is $(3, 0)$.

Step 2

We can find any vertical asymptotes be setting the denominator equal to zero and solving. Doing that for this function gives,

$$1 - x = 0 \qquad \rightarrow \qquad x = 1$$

So, we'll have a vertical asymptote at $x = 1$.

Step 3

For this equation the largest exponent of x in both the numerator and denominator is 1. Therefore, the horizontal asymptote for this problem is then the coefficient of the x in the numerator divided by the coefficient of the x in the denominator. Or,

$$y = \frac{-2}{-1} = 2$$

Step 4

From Step 2 we saw we only have one vertical asymptote and so we only have two regions to our graph : $x < 1$ and $x > 1$.

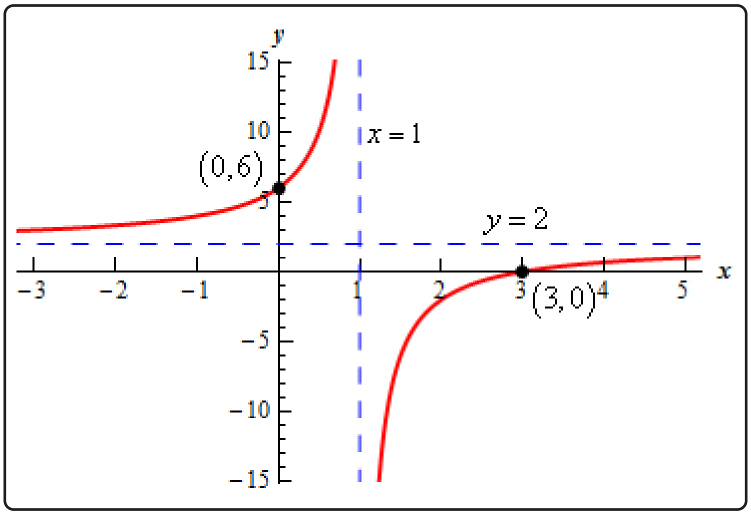
We'll need a point in each region to determine if it will be above or below the horizontal asymptote. Here are a couple of function evaluations for the points.

$$\begin{aligned} f(0) &= 6 \qquad \rightarrow \qquad (0,6) \\ f(3) &= 0 \qquad \rightarrow \qquad (3,0) \end{aligned}$$

Note that both of these are the intercepts we found in the first step. In this case they both just happened to be on either side of the vertical asymptote and so we could use these two points here.

Step 5

Here is a sketch of the function with the points found above. The vertical and horizontal asymptotes are indicated with blue dashed lines.



3. Sketch the graph of the following function. Clearly identify all intercepts and asymptotes.

$$f(x) = \frac{8}{x^2 + x - 6}$$

Step 1

Let's first find the intercepts for this function.

The y-intercept is the point $(0, f(0)) = \left(0, -\frac{4}{3}\right)$.

For the x-intercepts we set the numerator equal to zero and solve. However, in this case the numerator is a constant (8 specifically) and so can't ever be zero. Therefore, this function will have no x-intercepts.

Step 2

We can find any vertical asymptotes be setting the denominator equal to zero and solving. Doing that for this function gives,

$$x^2 + x - 6 = (x+3)(x-2) = 0 \qquad \to \qquad x = -3, \quad x = 2$$

So, we'll have two vertical asymptotes at $x = -3$ and $x = 2$.

Step 3

For this equation the largest exponent of x in the numerator is zero since the numerator is a constant. The largest exponent of x in the denominator is 2, which is larger than the largest exponent in the numerator, and so the x-axis will be the horizontal asymptote.

Step 4

From Step 2 we saw we only have two vertical asymptotes and so we have three regions to our graph : $x < -3$, $-3 < x < 2$ and $x > 2$.

We'll need a point in each region to determine if it will be above or below the horizontal asymptote.

Also, as we discussed in the notes, there are a couple of possible different behaviors in the middle region. To determine just what the behavior is we need to get a couple of points in this region. The best idea for points in the middle region is check a couple of points close to the vertical asymptotes we know if the edge is going to be above or below the horizontal asymptote.

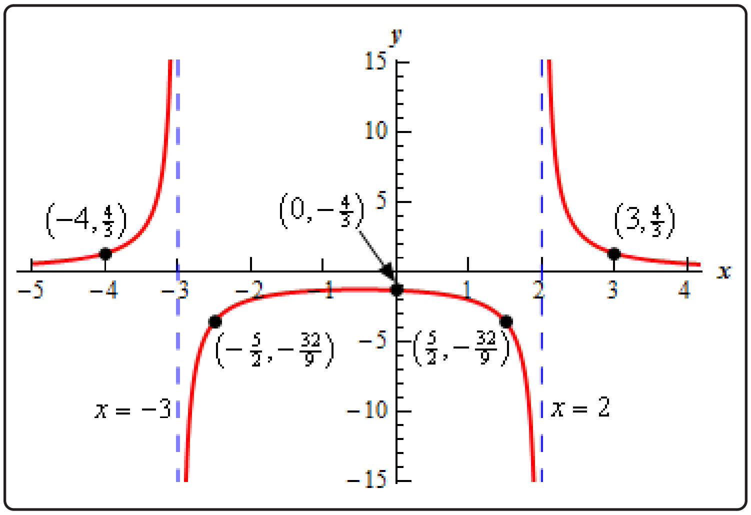
Here are some function evaluations for the points.

$$\begin{aligned} f\left(-4\right) &= \frac{4}{3} & \to \qquad & \left(-4, \frac{4}{3}\right) \\ f\left(-\frac{5}{2}\right) &= -\frac{32}{9} & \to \qquad & \left(-\frac{5}{2}, -\frac{32}{9}\right) \\ f\left(\frac{3}{2}\right) &= -\frac{32}{9} & \to \qquad & \left(\frac{3}{2}, -\frac{32}{9}\right) \\ f\left(3\right) &= \frac{4}{3} & \to \qquad & \left(3, \frac{4}{3}\right) \end{aligned}$$

From the second and third points we see that the curve in the middle region should be below the horizontal asymptote (x-axis for this problem) at both edges and so the curve will be completely below the horizontal asymptote in this whole region.

Step 5

Here is a sketch of the function with the points found above. The vertical asymptote is indicated with a blue dashed line and recall that the horizontal asymptote is just the x-axis.



4. Sketch the graph of the following function. Clearly identify all intercepts and asymptotes.

$$f(x) = \frac{4x^2 - 36}{x^2 - 2x - 8}$$

Step 1

Let's first find the intercepts for this function.

The y-intercept is the point $(0, f(0)) = \left(0, \frac{9}{2}\right)$.

For the x-intercepts we set the numerator equal to zero and solve. Doing that for this problem gives,

$$4x^2 - 36 = 0 \quad \rightarrow \quad x^2 = 9 \quad \rightarrow \quad x = \pm 3$$

So, the two x-intercepts for this problem are $(-3, 0)$ and $(3, 0)$.

Step 2

We can find any vertical asymptotes be setting the denominator equal to zero and solving. Doing that for this function gives,

$$x^2 - 2x - 8 = (x+2)(x-4) = 0 \quad \rightarrow \quad x = -2, \quad x = 4$$

So, we'll have two vertical asymptotes at $x = -2$ and $x = 4$.

Step 3

For this equation the largest exponent of x in both the numerator and denominator is 2. Therefore, the horizontal asymptote for this problem is then the coefficient of the x^2 in the numerator divided by the coefficient of the x^2 in the denominator. Or,

$$y = \frac{4}{1} = 4$$

Step 4

From Step 2 we saw we only have two vertical asymptotes and so we have three regions to our graph : $x < -2$, $-2 < x < 4$ and $x > 4$.

We'll need a point in each region to determine if it will be above or below the horizontal asymptote.

Also, as we discussed in the notes, there are a couple of possible different behaviors in the middle region. To determine just what the behavior is we need to get a couple of points in this region. The best idea for points in the middle region is check a couple of points close to the vertical asymptotes we know if the edge is going to be above or below the horizontal asymptote.

Here are some function evaluations for the points.

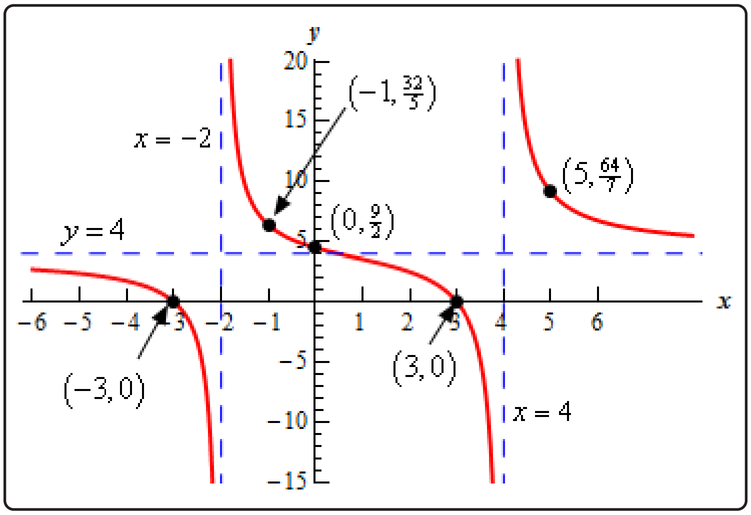
$$\begin{aligned} f(-3) &= 0 & &\to & &(-3,0) \\ f(-1) &= \frac{32}{5} & &\to & &\left(-1,\frac{32}{5}\right) \\ f(3) &= 0 & &\to & &(3,0) \\ f(5) &= \frac{64}{7} & &\to & &\left(5,\frac{64}{7}\right) \end{aligned}$$

From the second and third points we see that the curve in the middle region should be below the horizontal asymptote at the right edge and above the horizontal asymptote at the left edge and so the curve will cross the horizontal asymptote in this region.

Also note that we used the two x-intercepts here because they worked out to be good choices for points to use.

Step 5

Here is a sketch of the function with the points found above. The vertical and horizontal asymptotes are indicated with blue dashed lines.



5 Polynomial Functions

In this chapter we are going to take a more in depth look at polynomials. We've already solved and graphed second degree polynomials (i.e. quadratic equations/functions) and we now want to extend things out to more general polynomials. We will investigate dividing polynomials and determining where a polynomial equal to zero. After we know how to determine where a polynomial is zero we will take a look at how to get a rough sketch for a higher degree polynomial.

We will also be looking at Partial Fractions in this chapter. It doesn't really have anything to do with graphing polynomials but needed to be put somewhere and this chapter seemed like as good a place as any.

The following sections are the practice problems, with solutions, for this material.

5.1 Dividing Polynomials

1. Use long division to divide $3x^4 - 5x^2 + 3$ by $x + 2$.

Step 1

Let's first perform the long division. Just remember that we keep going until the remainder has degree that is strictly less that the degree of the polynomial we're dividing by, $x + 2$ in this case. The polynomial we're dividing by has degree one and so, in this case, we'll stop when the remainder is degree zero, *i.e.* a constant.

Here is the long division work for this problem.

$$\begin{array}{r l}
 & 3x^3 - 6x^2 + 7x - 14 \\
x+2 \,\big) & 3x^4 + 0x^3 - 5x^2 + 0x + 3 \\
 & -\underline{\left(-3x^4 + 6x^3\right)} \\
 & -6x^3 - 5x^2 + 0x + 3 \\
 & -\underline{\left(-6x^3 - 12x^2\right)} \\
 & 7x^2 + 0x + 3 \\
 & -\underline{\left(7x^2 + 14x\right)} \\
 & -14x + 3 \\
 & -\underline{\left(-14x - 28\right)} \\
 & 31
\end{array}$$

Step 2

We can put the answer in either of the two following forms.

$$\frac{3x^4 - 5x^2 + 3}{x + 2} = 3x^3 - 6x^2 + 7x - 14 + \frac{31}{x + 2}$$

$$3x^4 - 5x^2 + 3 = (x + 2)\left(3x^3 - 6x^2 + 7x - 14\right) + 31$$

Either answer is acceptable here although one may be more useful than the other depending on the application that is being done when you need to actually do the long division.

2. Use long division to divide $x^3 + 2x^2 - 3x + 4$ by $x - 7$.

Step 1

Let's first perform the long division. Just remember that we keep going until the remainder has degree that is strictly less that the degree of the polynomial we're dividing by, $x - 7$ in this case. The polynomial we're dividing by has degree one and so, in this case, we'll stop when the remainder is degree zero, *i.e.* a constant.

Here is the long division work for this problem.

$$\begin{array}{r} x^2 + 9x + 60 \\ x - 7 \overline{\smash{\big)}\, x^3 + 2x^2 - 3x + 4} \\ \underline{-\left(x^3 - 7x^2\right)} \qquad\qquad \\ 9x^2 - 3x + 4 \\ \underline{-\left(9x^2 - 63x\right)} \qquad \\ 60x + 4 \\ \underline{-\left(60x - 420\right)} \\ 424 \end{array}$$

Step 2

We can put the answer in either of the two following forms.

$$\frac{x^3 + 2x^2 - 3x + 4}{x - 7} = x^2 + 9x + 60 + \frac{424}{x - 7}$$

$$x^3 + 2x^2 - 3x + 4 = (x - 7)\left(x^2 + 9x + 60\right) + 424$$

Either answer is acceptable here although one may be more useful than the other depending on the application that is being done when you need to actually do the long division.

3. Use long division to divide $2x^5 + x^4 - 6x + 9$ by $x^2 - 3x + 1$.

Step 1

Let's first perform the long division. Just remember that we keep going until the remainder has degree that is strictly less that the degree of the polynomial we're dividing by, $x^2 - 3x + 1$ in this case. The polynomial we're dividing by has degree two and so, in this case, we'll stop when the remainder is degree one or zero.

Here is the long division work for this problem.

$$
\begin{array}{r}
2x^3 + 7x^2 + 19x + 50 \\
x^2 - 3x + 1 \enclose{longdiv}{2x^5 + x^4 + 0x^3 + 0x^2 - 6x + 9} \\
-\underline{\left(2x^5 - 6x^4 + 2x^3\right)} \\
7x^4 - 2x^3 + 0x^2 - 6x + 9 \\
-\underline{\left(7x^4 - 21x^3 + 7x^2\right)} \\
19x^3 - 7x^2 - 6x + 9 \\
-\underline{\left(19x^3 - 57x^2 + 19x\right)} \\
50x^2 - 25x + 9 \\
-\underline{\left(50x^2 - 150x + 50\right)} \\
125x - 41
\end{array}
$$

Step 2

We can put the answer in either of the two following forms.

$$\frac{2x^5 + x^4 - 6x + 9}{x^2 - 3x + 1} = 2x^3 + 7x^2 + 19x + 50 + \frac{125x - 41}{x^2 - 3x + 1}$$

$$2x^5 + x^4 - 6x + 9 = \left(x^2 - 3x + 1\right)\left(2x^3 + 7x^2 + 19x + 50\right) + 125x - 41$$

Either answer is acceptable here although one may be more useful than the other depending on the application that is being done when you need to actually do the long division.

4. Use synthetic division to divide $x^3 + x^2 + x + 1$ by $x + 9$.

Step 1

Here is the synthetic division. We'll leave it to you to check all the numbers.

$$\begin{array}{r|rrrr} -9 & 1 & 1 & 1 & 1 \\ & & -9 & 72 & -657 \\ \hline & 1 & -8 & 73 & -656 \end{array}$$

Step 2

The answer is then,

$$x^3 + x^2 + x + 1 = (x + 9)\left(x^2 - 8x + 73\right) - 656$$

Note that we only gave one form of the answer (unlike the first couple of problems) since this is often the form we need when using synthetic division and it is also the form that method is set up to give.

5. Use synthetic division to divide $7x^3 - 1$ by $x + 2$.

Step 1

Here is the synthetic division. We'll leave it to you to check all the numbers.

$$\begin{array}{r|rrrr} -2 & 7 & 0 & 0 & -1 \\ & & -14 & 28 & -56 \\ \hline & 7 & -14 & 28 & -57 \end{array}$$

Step 2

The answer is then,

$$7x^3 - 1 = (x + 2)\left(7x^2 - 14x + 28\right) - 57$$

Note that we only gave one form of the answer (unlike the first couple of problems) since this is often the form we need when using synthetic division and it is also the form that method is set up to give.

6. Use synthetic division to divide $5x^4 + x^2 - 8x + 2$ by $x - 4$.

Step 1

Here is the synthetic division. We'll leave it to you to check all the numbers.

$$\begin{array}{r|rrrrr} \underline{4} & 5 & 0 & 1 & -8 & 2 \\ & & 20 & 80 & 324 & 1264 \\ \hline & 5 & 20 & 81 & 316 & 1266 \end{array}$$

Step 2

The answer is then,

$$5x^4 + x^2 - 8x + 2 = (x - 4)\left(5x^3 + 20x^2 + 81x + 316\right) + 1266$$

Note that we only gave one form of the answer (unlike the first couple of problems) since this is often the form we need when using synthetic division and it is also the form that method is set up to give.

5.2 Zeroes/Roots of Polynomials

1. List all of the zeros of the following polynomial and give their multiplicities.

$$f(x) = 2x^2 + 13x - 7$$

Step 1

For this problem we'll first need to factor the polynomial.

$$f(x) = 2x^2 + 13x - 7 = (2x - 1)(x + 7)$$

From this we see that we have the two zeroes/roots : $x = \frac{1}{2}$ and $x = -7$.

Step 2

For the multiplicities just remember that the multiplicity of the zero/root is simply the exponent on the term that produces the zero/root. Therefore, the multiplicities of each zero/root is,

$$x = \frac{1}{2} : \text{ multiplicity } 1$$
$$x = -7 : \text{ multiplicity } 1$$

2. List all of the zeros of the following polynomial and give their multiplicities.

$$g(x) = x^6 - 3x^5 - 6x^4 + 10x^3 + 21x^2 + 9x = x(x-3)^2(x+1)^3$$

Step 1

For this problem the polynomial has already been factored and so all we need to do is get the zeroes/roots from the factored form.

The zeroes/roots of this polynomial are : $x = 0$, $x = 3$ and $x = -1$.

Step 2

For the multiplicities just remember that the multiplicity of the zero/root is simply the exponent on the term that produces the zero/root. Therefore, the multiplicities of each zero/root is,

$$\begin{aligned} x = 0 &: \text{ multiplicity 1} \\ x = 3 &: \text{ multiplicity 2} \\ x = -1 &: \text{ multiplicity 3} \end{aligned}$$

3. List all of the zeros of the following polynomial and give their multiplicities.

$$\begin{aligned} A(x) &= x^8 + 2x^7 - 29x^6 - 76x^5 + 199x^4 + 722x^3 + 261x^2 - 648x - 432 \\ &= (x+1)^2 (x-4)^2 (x-1)(x+3)^3 \end{aligned}$$

Step 1

For this problem the polynomial has already been factored and so all we need to do is get the zeroes/roots from the factored form.

The zeroes/roots of this polynomial are : $x = -1$, $x = 4$, $x = 1$ and $x = -3$.

Step 2

For the multiplicities just remember that the multiplicity of the zero/root is simply the exponent on the term that produces the zero/root. Therefore, the multiplicities of each zero/root is,

$$\begin{aligned} x = -1 &: \text{ multiplicity 2} \\ x = 4 &: \text{ multiplicity 2} \\ x = 1 &: \text{ multiplicity 1} \\ x = -3 &: \text{ multiplicity 3} \end{aligned}$$

4. $x = r$ is a root of the following polynomial. Find the other two roots and write the polynomial in fully factored form.

$$P(x) = x^3 - 6x^2 - 16x \; ; \; r = -2$$

Step 1

We know that $x = -2$ is a root of the polynomial and so we know that we can write the polynomial as,

$$P(x) = (x+2)\,Q(x)$$

Step 2

To find $Q(x)$ all we need to do is a quick synthetic division.

$$\begin{array}{r|rrrr} -2 & 1 & -6 & -16 & 0 \\ & & -2 & 16 & 0 \\ \hline & 1 & -8 & 0 & 0 \end{array}$$

From this we see that,

$$Q(x) = x^2 - 8x$$

Step 3

We can now write down $P(x)$ and it is simple enough to factor $Q(x)$.

$$P(x) = (x+2)\left(x^2 - 8x\right) = \boxed{x(x+2)(x-8)}$$

Step 4

Finally, from the factored for of $P(x)$ in the previous step we can see that the full list of roots/zeroes are,

$$\boxed{x = 0 \qquad x = -2 \qquad x = 8}$$

---

5. $x = r$ is a root of the following polynomial. Find the other two roots and write the polynomial in fully factored form.

$$P(x) = x^3 - 7x^2 - 6x + 72\ ;\ r = 4$$

Step 1

We know that $x = 4$ is a root of the polynomial and so we know that we can write the polynomial as,

$$P(x) = (x-4)\,Q(x)$$

Step 2

To find $Q(x)$ all we need to do is a quick synthetic division.

$$\begin{array}{r|rrrr} \underline{4} & 1 & -7 & -6 & 72 \\ & & 4 & -12 & -72 \\ \hline & 1 & -3 & -18 & 0 \end{array}$$

From this we see that,

$$Q(x) = x^2 - 3x - 18$$

Step 3

We can now write down $P(x)$ and it is simple enough to factor $Q(x)$.

$$P(x) = (x-4)\left(x^2 - 3x - 18\right) = \boxed{(x-4)(x-6)(x+3)}$$

Step 4

Finally, from the factored for of $P(x)$ in the previous step we can see that the full list of roots/zeroes are,

$$\boxed{x = 4 \qquad x = 6 \qquad x = -3}$$

---

6. $x = r$ is a root of the following polynomial. Find the other two roots and write the polynomial in fully factored form.

$$P(x) = 3x^3 + 16x^2 - 33x + 14\,;\, r = -7$$

Step 1

We know that $x = -7$ is a root of the polynomial and so we know that we can write the polynomial as,

$$P(x) = (x+7)\,Q(x)$$

Step 2

To find $Q(x)$ all we need to do is a quick synthetic division.

$$\begin{array}{r|rrrr} -7 & 3 & 16 & -33 & 14 \\ & & -21 & 35 & -14 \\ \hline & 3 & -5 & 2 & 0 \end{array}$$

From this we see that,

$$Q(x) = 3x^2 - 5x + 2$$

Step 3

We can now write down $P(x)$ and it is simple enough to factor $Q(x)$.

$$P(x) = (x+7)\left(3x^2 - 5x + 2\right) = \boxed{(x+7)(3x-2)(x-1)}$$

Step 4

Finally, from the factored for of $P(x)$ in the previous step we can see that the full list of roots/zeroes are,

$$\boxed{x = -7 \qquad x = \frac{2}{3} \qquad x = 1}$$

5.3 Graphing Polynomials

1. Sketch the graph of each of the following polynomial.

$$f(x) = x^3 - 2x^2 - 24x$$

Step 1

The first step is to determine the zeroes of the polynomial and the multiplicity of each zero. For this problem that means we'll need to start off factoring the polynomial.

$$f(x) = x^3 - 2x^2 - 24x = f(x) = x\left(x^2 - 2x - 24\right) = x(x-6)(x+4)$$

We have the following list of zeroes and multiplicities.

$x = -4$: multiplicity 1
$x = 0$: multiplicity 1
$x = 6$: multiplicity 1

Because the multiplicity of each of the zeroes is odd we know that each will correspond to an x-intercept that will cross the x-axis (as opposed to just touching the x-axis without actually crossing).

Step 2

The y-intercept for the function is : $(0, f(0)) = (0,0)$. In this case the point is also an x-intercept. This will happen on occasion so we'll need to deal with it.

Step 3

The coefficient of the 3^{rd} degree term is positive and so we know that the polynomial will increase without bound at the right end and decrease without bound at the left end.

Step 4

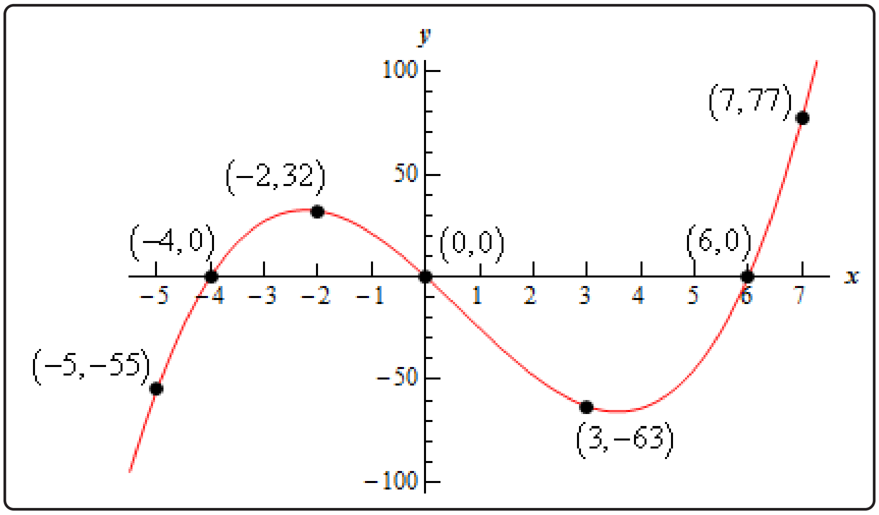
Next, we need to pick a couple of points to graph. We'll pick one point on the left and right end and it doesn't matter which points to pick here so just pick one close to the zero on either end. We'll also pick a point between each pair of zeroes and again it doesn't really matter which points to pick so we'll just pick points that are near the middle of each pair of zeroes.

Here are the function evaluations for this polynomial. We'll leave it to you to verify the evaluations.

$$f(-5) = -55 \qquad f(-2) = 32 \qquad f(3) = -63 \qquad f(7) = 77$$

Step 5

Finally, here is a quick sketch of the polynomial.



Note that your graph may not look quite like this one.

This is a computer generated graph and as such is very accurate. Using only Algebra techniques it can be quite difficult to approach this kind of accuracy. As long as the basic behavior of your graph is correct and you've got the extra points correctly graphed then you should have something that is close enough to the actual graph to work for a quick sketch.

For those interested, there are several techniques from a Calculus class that allow for a much more accurate sketch of the polynomial!

2. Sketch the graph of each of the following polynomial.

$$g(x) = -x^3 + 3x - 2 = -(x-1)^2(x+2)$$

Step 1

The first step is to determine the zeroes of the polynomial and the multiplicity of each zero. For this problem the polynomial has already been factored.

We have the following list of zeroes and multiplicities.

$$x = -2: \text{ multiplicity 1}$$
$$x = 1: \text{ multiplicity 2}$$

Because the multiplicity of $x = -2$ is odd we know that this point will correspond to an x-intercept that will cross the x-axis and $x = 1$ has an even multiplicity and so we know that this point will correspond to an x-intercept that will just touch the x-axis but not cross the x-axis.

Step 2

The y-intercept for the function is : $(0, g(0)) = (0, -2)$.

Step 3

The coefficient of the 3^{rd} degree term is negative and so we know that the polynomial will decrease without bound at the right end and increase without bound at the left end.

Step 4

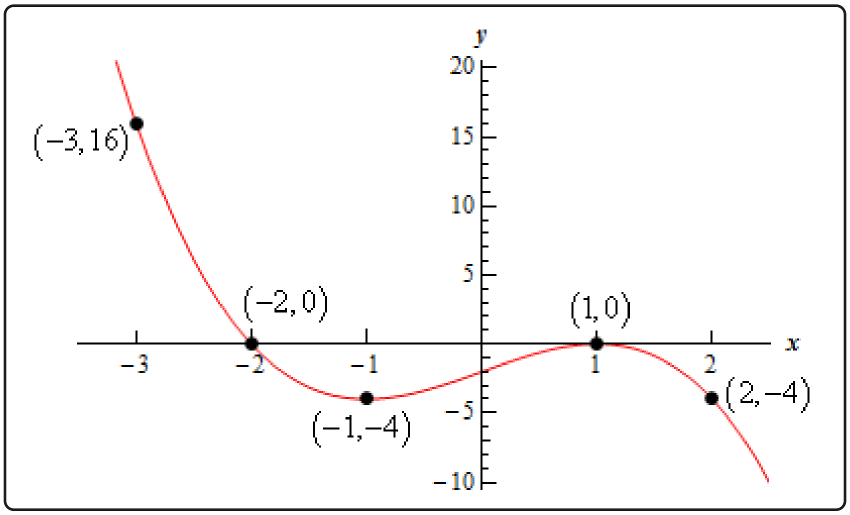
Next, we need to pick a couple of points to graph. We'll pick one point on the left and right end and it doesn't matter which points to pick here so just pick one close to the zero on either end. We'll also pick a point between each pair of zeroes and again it doesn't really matter which points to pick so we'll just pick points that are near the middle of each pair of zeroes.

Here are the function evaluations for this polynomial. We'll leave it to you to verify the evaluations.

$$g(-3) = 16 \qquad g(-1) = -4 \qquad g(2) = -4$$

Step 5

Finally, here is a quick sketch of the polynomial.



Note that your graph may not look quite like this one.

This is a computer generated graph and as such is very accurate. Using only Algebra techniques it can be quite difficult to approach this kind of accuracy. As long as the basic behavior of your graph is correct and you've got the extra points correctly graphed then you should have something that is close enough to the actual graph to work for a quick sketch.

For those interested, there are several techniques from a Calculus class that allow for a much more accurate sketch of the polynomial!

3. Sketch the graph of each of the following polynomial.

$$h(x) = x^4 + x^3 - 12x^2 + 4x + 16 = (x-2)^2(x+1)(x+4)$$

Step 1

The first step is to determine the zeroes of the polynomial and the multiplicity of each zero. For this problem the polynomial has already been factored.

We have the following list of zeroes and multiplicities.

$$x = -4 : \text{ multiplicity } 1$$
$$x = -1 : \text{ multiplicity } 1$$
$$x = 2 \; : \text{ multiplicity } 2$$

Because the multiplicity of $x = -4$ and $x = -1$ are odd we know that these point will correspond to x-intercepts that will cross the x-axis and $x = 2$ has an even multiplicity and so we know that this point will correspond to an x-intercept that will just touch the x-axis but not cross the x-axis.

Step 2

The y-intercept for the function is : $(0, h(0)) = (0, 16)$.

Step 3

The coefficient of the 4^{th} degree term is positive and so we know that the polynomial will increase without bound at both ends.

Step 4

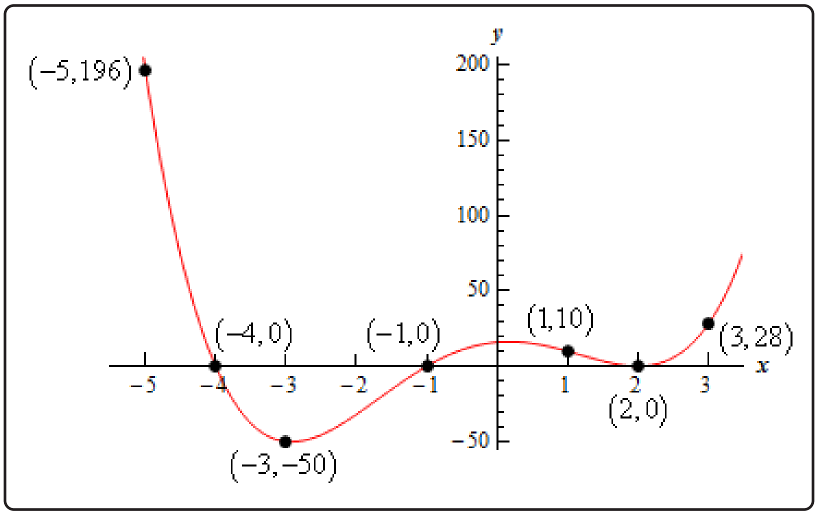
Next, we need to pick a couple of points to graph. We'll pick one point on the left and right end and it doesn't matter which points to pick here so just pick one close to the zero on either end. We'll also pick a point between each pair of zeroes and again it doesn't really matter which points to pick so we'll just pick points that are near the middle of each pair of zeroes.

Here are the function evaluations for this polynomial. We'll leave it to you to verify the evaluations.

$$h(-5) = 196 \qquad h(-3) = -50 \qquad h(1) = 10 \qquad h(3) = 28$$

Step 5

Finally, here is a quick sketch of the polynomial.



Note that your graph may not look quite like this one.

This is a computer generated graph and as such is very accurate. Using only Algebra techniques it can be quite difficult to approach this kind of accuracy. As long as the basic behavior of your graph is correct and you've got the extra points correctly graphed then you should have something that is close enough to the actual graph to work for a quick sketch.

For those interested, there are several techniques from a Calculus class that allow for a much more accurate sketch of the polynomial!

4. Sketch the graph of each of the following polynomial.

$$P(x) = x^5 - 12x^3 - 16x^2 = x^2(x+2)^2(x-4)$$

Step 1

The first step is to determine the zeroes of the polynomial and the multiplicity of each zero. For this problem the polynomial has already been factored.

We have the following list of zeroes and multiplicities.

$$x = -2: \text{ multiplicity 2}$$
$$x = 0: \text{ multiplicity 2}$$
$$x = 4: \text{ multiplicity 1}$$

Because the multiplicity of $x = 4$ is odd we know that this point will correspond to an x-intercept that will cross the x-axis while $x = -2$ and $x = 0$ have even multiplicity and so we know that these points will correspond to x-intercepts that will just touch the x-axis but not cross the x-axis.

Step 2

The y-intercept for the function is : $(0, P(0)) = (0, 0)$. In this case the point is also an x-intercept. This will happen on occasion so we'll need to deal with it.

Step 3

The coefficient of the 5^{th} degree term is positive and so we know that the polynomial will increase without bound at the right end and decrease without bound at the left end.

Step 4

Next, we need to pick a couple of points to graph. We'll pick one point on the left and right end and it doesn't matter which points to pick here so just pick one close to the zero on either end. We'll also pick a point between each pair of zeroes and again it doesn't really matter which points to pick so we'll just pick points that are near the middle of each pair of zeroes.

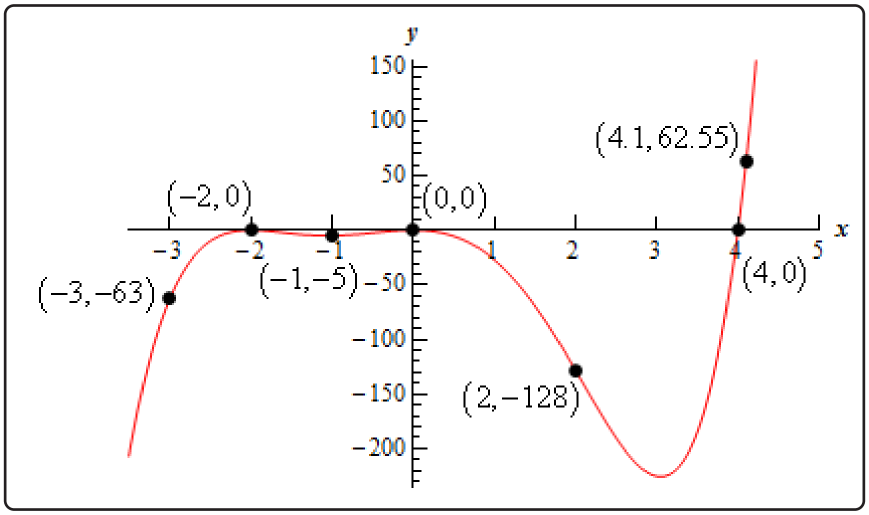
Here are the function evaluations for this polynomial. We'll leave it to you to verify the evaluations.

$$P(-3) = -63 \qquad P(-1) = -5 \qquad P(2) = -128 \qquad P(4.1) = 62.55$$

For the point at the right end point we chose 4.1 because the graph got very large very fast and this was the only way to keep the scale of the graph reasonable.

Step 5

Finally, here is a quick sketch of the polynomial.



Note that your graph may not look quite like this one.

This is a computer generated graph and as such is very accurate. Using only Algebra techniques it can be quite difficult to approach this kind of accuracy. That is especially true in this case with the portion of the graph between $x = 2$ and $x = 4$. You'd only get it that portion correct if you'd chosen to check the point at $x = 3$ which is not necessarily the obvious choice.

As long as the basic behavior of your graph is correct and you've got the extra points correctly graphed then you should have something that is close enough to the actual graph to work for a quick sketch.

For those interested, there are several techniques from a Calculus class that allow for a much more accurate sketch of the polynomial!

5.4 Finding Zeroes Of Polynomials

1. Find all the zeroes of the following polynomial.

$$f(x) = 2x^3 - 13x^2 + 3x + 18$$

Step 1

We'll need all the factors of 18 and 2.

$$\begin{aligned} 18 &: \pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18 \\ 2 &: \pm 1,\ \pm 2 \end{aligned}$$

Step 2

Here is a list of all possible rational zeroes for the polynomial.

$$\frac{\pm 1}{\pm 1} = \pm 1 \qquad \frac{\pm 2}{\pm 1} = \pm 2 \qquad \frac{\pm 3}{\pm 1} = \pm 3 \qquad \frac{\pm 6}{\pm 1} = \pm 6 \qquad \frac{\pm 9}{\pm 1} = \pm 9 \qquad \frac{\pm 18}{\pm 1} = \pm 18$$

$$\frac{\pm 1}{\pm 2} = \frac{\pm 1}{\pm 2} \qquad \frac{\pm 2}{\pm 2} = \pm 1 \qquad \frac{\pm 3}{\pm 2} = \frac{\pm 3}{\pm 2} \qquad \frac{\pm 6}{\pm 2} = \pm 3 \qquad \frac{\pm 9}{\pm 2} = \frac{\pm 9}{\pm 2} \qquad \frac{\pm 18}{\pm 2} = \pm 9$$

So, we have a total of 18 possible zeroes for the polynomial.

Step 3

We now need to start the synthetic division work. We'll start with the "small" integers first.

$$\begin{array}{r|rrrl} & 2 & -13 & 3 & 18 \\ \hline -1 & 2 & -15 & 18 & 0 = f(-1) = 0!! \end{array}$$

Okay we now know that $x = -1$ is a zero and we can write the polynomial as,

$$f(x) = 2x^3 - 13x^2 + 3x + 18 = (x+1)\left(2x^2 - 15x + 18\right)$$

Step 4

We could continue with this process however, we have a quadratic for the second factor and we can just factor this so the fully factored form of the polynomial is,

$$f(x) = 2x^3 - 13x^2 + 3x + 18 = (x+1)(2x-3)(x-6)$$

Step 5

From the fully factored form we get the following set of zeroes for the original polynomial.

$$\boxed{x = -1 \qquad x = \frac{3}{2} \qquad x = 6}$$

2. Find all the zeroes of the following polynomial.

$$P(x) = x^4 - 3x^3 - 5x^2 + 3x + 4$$

Step 1

We'll need all the factors of 4 and 1.

$$\begin{aligned} &4 : \pm 1, \pm 2, \pm 4 \\ &1 : \pm 1 \end{aligned}$$

Step 2

Here is a list of all possible rational zeroes for the polynomial.

$$\frac{\pm 1}{\pm 1} = \pm 1 \qquad \frac{\pm 2}{\pm 1} = \pm 2 \qquad \frac{\pm 4}{\pm 1} = \pm 4$$

So, we have a total of 6 possible zeroes for the polynomial.

Step 3

We now need to start the synthetic division work. We'll start with the "small" integers first.

$$\begin{array}{r|rrrrr} & 1 & -3 & -5 & 3 & 4 \\ \hline -1 & 1 & -4 & -1 & 4 & 0 = P(-1) = 0!! \end{array}$$

Okay we now know that $x = -1$ is a zero and we can write the polynomial as,

$$P(x) = x^4 - 3x^3 - 5x^2 + 3x + 4 = (x+1)\left(x^3 - 4x^2 - x + 4\right)$$

Step 4

So, now we need to continue the process using $Q(x) = x^3 - 4x^2 - x + 4$. The possible zeroes of this polynomial are the same as the original polynomial and so we won't write them back down.

Here's the synthetic division work for this $Q(x)$.

$$\begin{array}{r|rrrl} & 1 & -4 & -1 & 4 \\ \hline -1 & 1 & -5 & 4 & 0 = Q(-1) = 0!! \end{array}$$

Therefore, $x = -1$ is also a zero of $Q(x)$ and the factored form of $Q(x)$ is,

$$Q(x) = x^3 - 4x^2 - x + 4 = (x+1)\left(x^2 - 5x + 4\right)$$

This also means that the factored form of the original polynomial is now,

$$P(x) = x^4 - 3x^3 - 5x^2 + 3x + 4 = (x+1)(x+1)\left(x^2 - 5x + 4\right) = (x+1)^2\left(x^2 - 5x + 4\right)$$

Step 5

We're down to a quadratic polynomial and so we can and we can just factor this to get the fully factored form of the original polynomial. This is,

$$P(x) = x^4 - 3x^3 - 5x^2 + 3x + 4 = (x+1)^2 (x-4)(x-1)$$

Step 6

From the fully factored form we get the following set of zeroes for the original polynomial.

$$\boxed{x = -1 \ \text{(multiplicity 2)} \qquad x = 1 \qquad x = 4}$$

---

3. Find all the zeroes of the following polynomial.

$$A(x) = 2x^4 - 7x^3 - 2x^2 + 28x - 24$$

Step 1

We'll need all the factors of -24 and 2.

$$\begin{aligned} -24 &: \pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24 \\ 2 &: \pm1, \pm2 \end{aligned}$$

Step 2

Here is a list of all possible rational zeroes for the polynomial.

$$\frac{\pm1}{\pm1} = \pm1 \qquad \frac{\pm2}{\pm1} = \pm2 \qquad \frac{\pm3}{\pm1} = \pm3 \qquad \frac{\pm4}{\pm1} = \pm4 \qquad \frac{\pm6}{\pm1} = \pm6 \qquad \frac{\pm8}{\pm1} = \pm8$$

$$\frac{\pm12}{\pm1} = \pm12 \qquad \frac{\pm24}{\pm1} = \pm24$$

$$\frac{\pm1}{\pm2} = \frac{\pm1}{\pm2} \qquad \frac{\pm2}{\pm2} = \pm1 \qquad \frac{\pm3}{\pm2} = \frac{\pm3}{\pm2} \qquad \frac{\pm4}{\pm2} = \pm2 \qquad \frac{\pm6}{\pm2} = \pm3 \qquad \frac{\pm8}{\pm2} = \pm4$$

$$\frac{\pm12}{\pm2} = \pm6 \qquad \frac{\pm24}{\pm2} = \pm12$$

So, we have a total of 20 possible zeroes for the polynomial.

Step 3

We now need to start the synthetic division work. We'll start with the "small" integers first.

$$\begin{array}{r|rrrrl} & 2 & -7 & -2 & 28 & -24 \\ \hline -1 & 2 & -9 & 7 & 21 & -45 = A(-1) \neq 0 \\ 1 & 2 & -5 & -7 & 21 & -3 = A(1) \neq 0 \\ -2 & 2 & -11 & 20 & -12 & 0 = A(-2) = 0!! \end{array}$$

Okay we now know that $x = -2$ is a zero and we can write the polynomial as,

$$A(x) = 2x^4 - 7x^3 - 2x^2 + 28x - 24 = (x+2)\left(2x^3 - 11x^2 + 20x - 12\right)$$

Step 4

So, now we need to continue the process using $Q(x) = 2x^3 - 11x^2 + 20x - 12$. Here is a list of all possible zeroes of $Q(x)$.

$$\frac{\pm 1}{\pm 1} = \pm 1 \qquad \frac{\pm 2}{\pm 1} = \pm 2 \qquad \frac{\pm 3}{\pm 1} = \pm 3 \qquad \frac{\pm 4}{\pm 1} = \pm 4 \qquad \frac{\pm 6}{\pm 1} = \pm 6 \qquad \frac{\pm 12}{\pm 1} = \pm 12$$

$$\frac{\pm 1}{\pm 2} = \frac{\pm 1}{\pm 2} \qquad \frac{\pm 2}{\pm 2} = \pm 1 \qquad \frac{\pm 3}{\pm 2} = \frac{\pm 3}{\pm 2} \qquad \frac{\pm 4}{\pm 2} = \pm 2 \qquad \frac{\pm 6}{\pm 2} = \pm 3 \qquad \frac{\pm 12}{\pm 2} = \pm 6$$

So we have a list of 16 possible zeroes, but note that we've already proved that ± 1 can't be zeroes of the original polynomial and so can't be zeroes of $Q(x)$ either.

Here's the synthetic division work for this $Q(x)$.

$$\begin{array}{r|llll} & 2 & -11 & 20 & -12 \\ \hline -2 & 2 & -15 & 50 & -112 = Q(-2) \neq 0 \\ 2 & 2 & -7 & 6 & 0 = Q(2) = 0!! \end{array}$$

Therefore, $x = 2$ is a zero of $Q(x)$ and the factored form of $Q(x)$ is,

$$Q(x) = 2x^3 - 11x^2 + 20x - 12 = (x-2)\left(2x^2 - 7x + 6\right)$$

This also means that the factored form of the original polynomial is now,

$$A(x) = 2x^4 - 7x^3 - 2x^2 + 28x - 24 = (x+2)(x-2)\left(2x^2 - 7x + 6\right)$$

Step 5

We're down to a quadratic polynomial and so we can and we can just factor this to get the fully factored form of the original polynomial. This is,

$$\begin{aligned} A(x) = 2x^4 - 7x^3 - 2x^2 + 28x - 24 &= (x+2)(x-2)(x-2)(2x-3) \\ &= (x+2)(x-2)^2(2x-3) \end{aligned}$$

Step 6

From the fully factored form we get the following set of zeroes for the original polynomial.

$$\boxed{x = -2 \qquad x = \frac{3}{2} \qquad x = 2 \;\; \text{(multiplicity 2)}}$$

4. Find all the zeroes of the following polynomial.

$$g(x) = 8x^5 + 36x^4 + 46x^3 + 7x^2 - 12x - 4$$

Step 1

We'll need all the factors of -4 and 8.

$$\begin{aligned} -4 &: \pm 1, \pm 2, \pm 4 \\ 8 &: \pm 1, \pm 2, \pm 4, \pm 8 \end{aligned}$$

Step 2

Here is a list of all possible rational zeroes for the polynomial.

$$\frac{\pm 1}{\pm 1} = \pm 1 \qquad \frac{\pm 2}{\pm 1} = \pm 2 \qquad \frac{\pm 4}{\pm 1} = \pm 4$$

$$\frac{\pm 1}{\pm 2} = \frac{\pm 1}{\pm 2} \qquad \frac{\pm 2}{\pm 2} = \pm 1 \qquad \frac{\pm 4}{\pm 2} = \pm 2$$

$$\frac{\pm 1}{\pm 4} = \frac{\pm 1}{\pm 4} \qquad \frac{\pm 2}{\pm 4} = \frac{\pm 1}{\pm 2} \qquad \frac{\pm 4}{\pm 4} = \pm 1$$

$$\frac{\pm 1}{\pm 8} = \frac{\pm 1}{\pm 8} \qquad \frac{\pm 2}{\pm 8} = \frac{\pm 1}{\pm 4} \qquad \frac{\pm 4}{\pm 8} = \frac{\pm 1}{\pm 2}$$

So, we have a total of 12 possible zeroes for the polynomial.

Step 3

We now need to start the synthetic division work. We'll start with the "small" integers first.

$$\begin{array}{r|rrrrrl} & 8 & 36 & 46 & 7 & -12 & -4 \\ \hline -1 & 8 & 28 & 18 & -11 & -1 & -3 = g(-1) \neq 0 \\ 1 & 8 & 44 & 90 & 97 & 85 & 81 = g(1) \neq 0 \end{array}$$

Okay, notice that we have opposite signs for the two function evaluations listed above. Recall that means that we know we have a zero somewhere between them.

So, let's take a look at some of the fractions from our list and give them a try in the synthetic division table. We'll start with the fractions with the smallest denominators.

$$\begin{array}{r|rrrrrl} & 8 & 36 & 46 & 7 & -12 & -4 \\ \hline -\frac{1}{2} & 8 & 32 & 30 & -8 & -8 & 0 = g(-1) = 0!! \end{array}$$

We now know that $x = -\frac{1}{2}$ is a zero and we can write the polynomial as,

$$g(x) = 8x^5 + 36x^4 + 46x^3 + 7x^2 - 12x - 4 = \left(x + \frac{1}{2}\right)\left(8x^4 + 32x^3 + 30x^2 - 8x - 8\right)$$

Step 4

So, now we need to continue the process using $Q(x) = 8x^4 + 32x^3 + 30x^2 - 8x - 8$. Here is a list of all possible zeroes of $Q(x)$.

$$\frac{\pm 1}{\pm 1} = \pm 1 \qquad \frac{\pm 2}{\pm 1} = \pm 2 \qquad \frac{\pm 4}{\pm 1} = \pm 4 \qquad \frac{\pm 8}{\pm 1} = \pm 8$$

$$\frac{\pm 1}{\pm 2} = \frac{\pm 1}{\pm 2} \qquad \frac{\pm 2}{\pm 2} = \pm 1 \qquad \frac{\pm 4}{\pm 2} = \pm 2 \qquad \frac{\pm 8}{\pm 2} = \pm 4$$

$$\frac{\pm 1}{\pm 4} = \frac{\pm 1}{\pm 4} \qquad \frac{\pm 2}{\pm 4} = \frac{\pm 1}{\pm 2} \qquad \frac{\pm 4}{\pm 4} = \pm 1 \qquad \frac{\pm 8}{\pm 4} = \pm 2$$

$$\frac{\pm 1}{\pm 8} = \frac{\pm 1}{\pm 8} \qquad \frac{\pm 2}{\pm 8} = \frac{\pm 1}{\pm 4} \qquad \frac{\pm 4}{\pm 8} = \frac{\pm 1}{\pm 2} \qquad \frac{\pm 8}{\pm 8} = \pm 1$$

So we have a list of 14 possible zeroes (lots of repeats), but note that we've already proved that ± 1 can't be zeroes of the original polynomial and so can't be zeroes of $Q(x)$ either.

Here's the synthetic division work for this $Q(x)$.

$$\begin{array}{r|rrrrl} & 8 & 32 & 30 & -8 & -8 \\ \hline -2 & 8 & 16 & -2 & -4 & 0 = Q(-2) = 0!! \end{array}$$

Therefore, $x = -2$ is a zero of $Q(x)$ and the factored form of $Q(x)$ is,

$$Q(x) = 8x^4 + 32x^3 + 30x^2 - 8x - 8 = (x+2)\left(8x^3 + 16x^2 - 2x - 4\right)$$

This also means that the factored form of the original polynomial is now,

$$g(x) = 8x^5 + 36x^4 + 46x^3 + 7x^2 - 12x - 4 = \left(x + \frac{1}{2}\right)(x+2)\left(8x^3 + 16x^2 - 2x - 4\right)$$

Step 5

So, it looks like we need to continue with the synthetic division. This time we'll do it on the polynomial $P(x) = 8x^3 + 16x^2 - 2x - 4$.

The possible zeroes of this are the same as the original polynomial and so we won't write them down here. Again, however, we've already proved that ± 1 can't be zeroes of the original polynomial and so can't be zeroes of $P(x)$ either.

Here is the synthetic division for $P(x)$.

$$\begin{array}{r|rrrl} & 8 & 16 & -2 & -4 \\ \hline -2 & 8 & 0 & -2 & 0 = P(-2) = 0!! \end{array}$$

Therefore, $x = -2$ is a zero of $P(x)$ and the factored form of $P(x)$ is,

$$P(x) = 8x^3 + 16x^2 - 2x - 4 = (x+2)\left(8x^2 - 2\right) = 8(x+2)\left(x^2 - \frac{1}{4}\right)$$

We factored an 8 out of the quadratic to make it a little easier for the next step.

The factored form of the original polynomial is now,

$$g(x) = 8x^5 + 36x^4 + 46x^3 + 7x^2 - 12x - 4 = 8\left(x + \frac{1}{2}\right)(x+2)^2\left(x^2 - \frac{1}{4}\right)$$

Step 6

We're down to a quadratic polynomial and so we can and we can just factor this to get the fully factored form of the original polynomial. This is,

$$\begin{aligned} g(x) = 8x^5 + 36x^4 + 46x^3 + 7x^2 - 12x - 4 &= 8\left(x + \frac{1}{2}\right)(x+2)^2\left(x + \frac{1}{2}\right)\left(x - \frac{1}{2}\right) \\ &= 8\left(x + \frac{1}{2}\right)^2(x+2)^2\left(x - \frac{1}{2}\right) \end{aligned}$$

Step 7

From the fully factored form we get the following set of zeroes for the original polynomial.

$$\boxed{x = -\frac{1}{2} \text{ (multiplicity 2)} \qquad x = -2 \text{ (multiplicity 2)} \qquad x = \frac{1}{2}}$$

Note that this problem was VERY long and messy. The point of this problem was really just to illustrate just how long and messy these can get. The moral, if there is one, is that we generally sit back and really hope that we don't have to work these kinds of problems on a regular basis. They are just too long and it's too easy to make a mistake with them.

5.5 Partial Fractions

1. Determine the partial fraction decomposition of each of the following expression.

$$\frac{17x-53}{x^2-2x-15}$$

Step 1

The first step is to determine the form of the partial fraction decomposition. However, in order to do that we first need to factor the denominator as much as possible. Doing this gives,

$$\frac{17x-53}{(x-5)(x+3)}$$

Okay, we can now see that the partial fraction decomposition is,

$$\frac{17x-53}{x^2-2x-15}=\frac{A}{x-5}+\frac{B}{x+3}$$

Step 2

The LCD for this expression is $(x-5)(x+3)$. Adding the two terms back up gives,

$$\frac{17x-53}{x^2-2x-15}=\frac{A(x+3)+B(x-5)}{(x-5)(x+3)}$$

Step 3

Setting the numerators equal gives,

$$17x-53=A(x+3)+B(x-5)$$

Step 4

Now all we need to do is pick "good" values of x to determine the constants. Here is that work.

$$\begin{array}{lcc} x=5: & 32=8A & \multirow{2}{*}{$\rightarrow$} & A=4 \\ x=-3: & -104=-8B & & B=13 \end{array}$$

Step 5

The partial fraction decomposition is then,

$$\boxed{\frac{17x-53}{x^2-2x-15}=\frac{4}{x-5}+\frac{13}{x+3}}$$

---

2. Determine the partial fraction decomposition of each of the following expression.

$$\frac{34-12x}{3x^2-10x-8}$$

Step 1

The first step is to determine the form of the partial fraction decomposition. However, in order to do that we first need to factor the denominator as much as possible. Doing this gives,

$$\frac{34-12x}{(3x+2)(x-4)}$$

Okay, we can now see that the partial fraction decomposition is,

$$\frac{34-12x}{3x^2-10x-8}=\frac{A}{3x+2}+\frac{B}{x-4}$$

Step 2

The LCD for this expression is $(3x+2)(x-4)$. Adding the two terms back up gives,

$$\frac{34-12x}{3x^2-10x-8}=\frac{A(x-4)+B(3x+2)}{(3x+2)(x-4)}$$

Step 3

Setting the numerators equal gives,

$$34-12x=A(x-4)+B(3x+2)$$

Step 4

Now all we need to do is pick "good" values of x to determine the constants. Here is that work.

$$\begin{array}{lll} x = 4: & -14 = 14B & \\ x = -\frac{2}{3}: & 42 = -\frac{14}{3}A & \end{array} \quad \to \quad \begin{array}{l} A = -9 \\ B = -1 \end{array}$$

Step 5

The partial fraction decomposition is then,

$$\boxed{\frac{34 - 12x}{3x^2 - 10x - 8} = -\frac{9}{3x+2} - \frac{1}{x-4}}$$

3. Determine the partial fraction decomposition of each of the following expression.

$$\frac{125 + 4x - 9x^2}{(x-1)(x+3)(x+4)}$$

Step 1

The first step is to determine the form of the partial fraction decomposition. For this problem the partial fraction decomposition is,

$$\frac{125 + 4x - 9x^2}{(x-1)(x+3)(x+4)} = \frac{A}{x-1} + \frac{B}{x+3} + \frac{C}{x+4}$$

Step 2

The LCD for this expression is $(x-1)(x+3)(x+4)$. Adding the terms back up gives,

$$\frac{125 + 4x - 9x^2}{(x-1)(x+3)(x+4)} = \frac{A(x+3)(x+4) + B(x-1)(x+4) + C(x-1)(x+3)}{(x-1)(x+3)(x+4)}$$

Step 3

Setting the numerators equal gives,

$$125 + 4x - 9x^2 = A(x+3)(x+4) + B(x-1)(x+4) + C(x-1)(x+3)$$

Step 4

Now all we need to do is pick “good” values of x to determine the constants. Here is that work.

$$\begin{array}{lcc} x=-4: & -35=5C & & A=6 \\ x=-3: & 32=-4B & \to & B=-8 \\ x=1: & 120=20A & & C=-7 \end{array}$$

Step 5

The partial fraction decomposition is then,

$$\boxed{\frac{125+4x-9x^2}{(x-1)(x+3)(x+4)}=\frac{6}{x-1}-\frac{8}{x+3}-\frac{7}{x+4}}$$

4. Determine the partial fraction decomposition of each of the following expression.

$$\frac{10x+35}{(x+4)^2}$$

Step 1

The first step is to determine the form of the partial fraction decomposition. For this problem the partial fraction decomposition is,

$$\frac{10x+35}{(x+4)^2}=\frac{A}{x+4}+\frac{B}{(x+4)^2}$$

Step 2

The LCD for this expression is $(x+4)^2$. Adding the terms back up gives,

$$\frac{10x+35}{(x+4)^2}=\frac{A(x+4)+B}{(x+4)^2}$$

Step 3

Setting the numerators equal gives,

$$10x + 35 = A\left(x + 4\right) + B$$

Step 4

For this problem we can pick a "good" value of x to determine only one of the constants. Here is that work.

$$x = -4: \qquad -5 = B \qquad \to \qquad B = -5$$

Step 5

To get the remaining constant we can use any value of x and plug that along with the value of B we found in the previous step into the equation from Step 3.

It really doesn't matter what value of x we pick as long as it isn't $x = -4$ since we used that in the previous step. The idea here is to pick a value of x that won't create "large" or "messy" numbers, if possible. Good choices are often $x = 0$ or $x = 1$, provided they weren't used in the previous step of course.

For this problem $x = 0$ seems to be a good choice. Here is the work for this step.

$$\begin{aligned} 10\left(0\right) + 35 &= A\left(0 + 4\right) + \left(-5\right) \\ 35 &= 4A - 5 \\ 40 &= 4A \qquad \Rightarrow \qquad A = 10 \end{aligned}$$

Step 6

The partial fraction decomposition is then,

$$\boxed{\frac{10x + 35}{\left(x + 4\right)^2} = \frac{10}{x + 4} - \frac{5}{\left(x + 4\right)^2}}$$

5. Determine the partial fraction decomposition of each of the following expression.

$$\frac{6x + 5}{\left(2x - 1\right)^2}$$

Step 1

The first step is to determine the form of the partial fraction decomposition. For this problem the partial fraction decomposition is,

$$\frac{6x+5}{(2x-1)^2} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2}$$

Step 2

The LCD for this expression is $(2x-1)^2$. Adding the terms back up gives,

$$\frac{6x+5}{(2x-1)^2} = \frac{A(2x-1)+B}{(2x-1)^2}$$

Step 3

Setting the numerators equal gives,

$$6x+5 = A(2x-1)+B$$

Step 4

For this problem we can pick a "good" value of x to determine only one of the constants. Here is that work.

$$x = \frac{1}{2}: \qquad 8 = B \qquad \to \qquad B = 8$$

Step 5

To get the remaining constant we can use any value of x and plug that along with the value of B we found in the previous step into the equation from Step 3.

It really doesn't matter what value of x we pick as long as it isn't $x = \frac{1}{2}$ since we used that in the previous step. The idea here is to pick a value of x that won't create "large" or "messy" numbers, if possible. Good choices are often $x = 0$ or $x = 1$, provided they weren't used in the previous step of course.

For this problem $x = 0$ seems to be a good choice. Here is the work for this step.

$$\begin{aligned} 6(0) + 5 &= A(2(0) - 1) + 8 \\ 5 &= -A + 8 \\ -3 &= -A \qquad \Rightarrow \qquad A = 3 \end{aligned}$$

Step 6

The partial fraction decomposition is then,

$$\boxed{\frac{6x+5}{(2x-1)^2} = \frac{3}{2x-1} + \frac{8}{(2x-1)^2}}$$

6. Determine the partial fraction decomposition of each of the following expression.

$$\frac{7x^2-17x+38}{(x+6)(x-1)^2}$$

Step 1

The first step is to determine the form of the partial fraction decomposition. For this problem the partial fraction decomposition is,

$$\frac{7x^2-17x+38}{(x+6)(x-1)^2}=\frac{A}{x+6}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

Step 2

The LCD for this expression is $(x+6)(x-1)^2$. Adding the terms back up gives,

$$\frac{7x^2-17x+38}{(x+6)(x-1)^2}=\frac{A(x-1)^2+B(x+6)(x-1)+C(x+6)}{(x+6)(x-1)^2}$$

Step 3

Setting the numerators equal gives,

$$7x^2-17x+38=A(x-1)^2+B(x+6)(x-1)+C(x+6)$$

Step 4

For this problem we can pick "good" values of x to determine only two of the three constants. Here is that work.

$$\begin{array}{lll} x=-6: & 392=A(-7)^2=49A & \\ & & \rightarrow \\ x=1: & 28=7C & \end{array} \qquad \begin{array}{l} A=8 \\ C=4 \end{array}$$

Step 5

To get the remaining constant we can use any value of x and plug that along with the values of A and C we found in the previous step into the equation from Step 3.

It really doesn't matter what value of x we pick as long as it isn't $x=-6$ or $x=1$ since we used those in the previous step. The idea here is to pick a value of x that won't create "large" or "messy" numbers, if possible. Good choices are often $x = 0$ or $x = 1$ (which we can't use for this problem as noted above), provided they weren't used in the previous step of course.

For this problem $x = 0$ seems to be a good choice. Here is the work for this step.

$$\begin{aligned}38 &= (8)(-1)^2 + B(6)(-1) + (4)(6)\\ 38 &= 32 - 6B\\ 6 &= -6B \qquad \Rightarrow \qquad B = -1\end{aligned}$$

Step 6

The partial fraction decomposition is then,

$$\boxed{\frac{7x^2 - 17x + 38}{(x+6)(x-1)^2} = \frac{8}{x+6} - \frac{1}{x-1} + \frac{4}{(x-1)^2}}$$

7. Determine the partial fraction decomposition of each of the following expression.

$$\frac{4x^2 - 22x + 7}{(2x+3)(x-2)^2}$$

Step 1

The first step is to determine the form of the partial fraction decomposition. For this problem the partial fraction decomposition is,

$$\frac{4x^2 - 22x + 7}{(2x+3)(x-2)^2} = \frac{A}{2x+3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

Step 2

The LCD for this expression is $(2x+3)(x-2)^2$. Adding the terms back up gives,

$$\frac{4x^2 - 22x + 7}{(2x+3)(x-2)^2} = \frac{A(x-2)^2 + B(2x+3)(x-2) + C(2x+3)}{(2x+3)(x-2)^2}$$

Step 3

Setting the numerators equal gives,

$$4x^2 - 22x + 7 = A(x-2)^2 + B\left(2x+3\right)\left(x-2\right) + C\left(2x+3\right)$$

Step 4

For this problem we can pick "good" values of x to determine only two of the three constants. Here is that work.

$$\begin{array}{lll} x = -\dfrac{3}{2}: & 49 = A\left(-\dfrac{7}{2}\right)^2 = \dfrac{49}{4}A & \quad\to\quad A = 4 \\ x = 2: & -21 = 7C & \quad\quad C = -3 \end{array}$$

Step 5

To get the remaining constant we can use any value of x and plug that along with the values of A and C we found in the previous step into the equation from Step 3.

It really doesn't matter what value of x we pick as long as it isn't $x = -\frac{3}{2}$ or $x = 2$ since we used those in the previous step. The idea here is to pick a value of x that won't create "large" or "messy" numbers, if possible. Good choices are often $x = 0$ or $x = 1$, provided they weren't used in the previous step of course.

For this problem $x = 0$ seems to be a good choice. Here is the work for this step.

$$\begin{aligned} 7 &= \left(4\right)\left(-2\right)^2 + B\left(3\right)\left(-2\right) + \left(-3\right)\left(3\right) \\ 7 &= 7 - 6B \\ 0 &= -6B \qquad \Rightarrow \qquad B = 0 \end{aligned}$$

In this case one of the constants ended up being zero. This happens on occasion but there is no way, in general, to know ahead of time that was going to happen so don't worry about it. If it turns out one of the constants is zero then we'll figure that out when we do the work.

Step 6

The partial fraction decomposition is then,

$$\boxed{\frac{4x^2 - 22x + 7}{(2x+3)(x-2)^2} = \frac{4}{2x+3} - \frac{3}{(x-2)^2}}$$

---

8. Determine the partial fraction decomposition of each of the following expression.

$$\frac{3x^2 + 7x + 28}{x\left(x^2 + x + 7\right)}$$

Step 1

The first step is to determine the form of the partial fraction decomposition. For this problem the partial fraction decomposition is,

$$\frac{3x^2 + 7x + 28}{x\left(x^2 + x + 7\right)} = \frac{A}{x} + \frac{Bx + C}{x^2 + x + 7}$$

Step 2

The LCD for this expression is $x\left(x^2 + x + 7\right)$. Adding the terms back up gives,

$$\frac{3x^2 + 7x + 28}{x\left(x^2 + x + 7\right)} = \frac{A\left(x^2 + x + 7\right) + (Bx + C)(x)}{x\left(x^2 + x + 7\right)}$$

Step 3

Setting the numerators equal gives,

$$3x^2 + 7x + 28 = A\left(x^2 + x + 7\right) + (Bx + C)(x)$$

Step 4

Because we have an unfactorable quadratic equation here the method we used in the first problems from this section won't work. So, we will need to multiply everything out and collect like terms.

$$\begin{aligned} 3x^2 + 7x + 28 &= Ax^2 + Ax + 7A + Bx^2 + Cx \\ &= (A+B)\,x^2 + (A+C)\,x + 7A \end{aligned}$$

Step 5

We now need to set coefficients equal. Remember this just means setting the coefficient of the x^2 on both sides equal and similarly for the coefficients of the x and the constants. Doing this gives,

$$\begin{aligned} A + B &= 3 \\ A + C &= 7 \\ 7A &= 28 \end{aligned}$$

Step 6

As mentioned in the notes this is a system of equations that we really haven't talked about how to solve in general, but that is not a real problem. From the third equation we can see that we must have $A = 4$.

Once we have this we only need to plug that into the first two equations to determine the values of B and C. Here is that work,

$$\begin{aligned} 4 + B &= 3 \\ 4 + C &= 7 \end{aligned} \qquad \to \qquad \begin{aligned} B &= -1 \\ C &= 3 \end{aligned}$$

Step 7

The partial fraction decomposition is then,

$$\boxed{\frac{3x^2 + 7x + 28}{x\,(x^2 + x + 7)} = \frac{4}{x} + \frac{3 - x}{x^2 + x + 7}}$$

---

9. Determine the partial fraction decomposition of each of the following expression.

$$\frac{4x^3 + 16x + 7}{\left(x^2 + 4\right)^2}$$

Step 1

The first step is to determine the form of the partial fraction decomposition. For this problem the partial fraction decomposition is,

$$\frac{4x^3+16x+7}{\left(x^2+4\right)^2}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{\left(x^2+4\right)^2}$$

Step 2

The LCD for this expression is $\left(x^2+4\right)^2$. Adding the terms back up gives,

$$\frac{4x^3+16x+7}{\left(x^2+4\right)^2}=\frac{(Ax+B)\left(x^2+4\right)+Cx+D}{\left(x^2+4\right)^2}$$

Step 3

Setting the numerators equal gives,

$$4x^3+16x+7=(Ax+B)\left(x^2+4\right)+Cx+D$$

Step 4

Because we have an unfactorable quadratic equation here the method we used in the first problems from this section won't work. So, we will need to multiply everything out and collect like terms.

$$\begin{aligned}4x^3+16x+7&=Ax^3+4Ax+Bx^2+4B+Cx+D\\&=Ax^3+Bx^2+(4A+C)\,x+4B+D\end{aligned}$$

Step 5

We now need to set coefficients equal. Remember this just means setting the coefficient of the x^3 on both sides equal and similarly for the coefficients of the x^2, the x and the constants. Doing this gives,

$$\begin{aligned} A &= 4 \\ B &= 0 \\ 4A + C &= 16 \\ 4B + D &= 7 \end{aligned}$$

Step 6

As mentioned in the notes this is a system of equations that we really haven't talked about how to solve in general, but that is not a real problem. From the first two equations we can see that we must have $A = 4$ and $B = 0$.

Once we have this we only need to plug that into the last two equations to determine the values of C and D. Here is that work,

$$\begin{aligned} 4(4) + C &= 16 \\ 4(0) + D &= 7 \end{aligned} \qquad \to \qquad \begin{aligned} C &= 0 \\ D &= 7 \end{aligned}$$

In this case two of the constants ended up being zero. This happens on occasion but there is no way, in general, to know ahead of time that was going to happen so don't worry about it. If it turns out one, or more, of the constants are zero then we'll figure that out when we do the work.

Step 7

The partial fraction decomposition is then,

$$\boxed{\frac{4x^3 + 16x + 7}{\left(x^2+4\right)^2} = \frac{4x}{x^2+4} + \frac{7}{\left(x^2+4\right)^2}}$$

6 Exponential and Logarithm Functions

In this chapter we are going to look at exponential and logarithm functions. Both of these functions are very important and need to be understood by anyone who is going on to later math courses. These functions also have applications in science, engineering, and business to name a few areas. In fact, these functions can show up in just about any field that uses even a small degree of mathematics.

Many students find both of these functions, especially logarithm functions, difficult to deal with. This is probably because they are so different from any of the other functions that they've looked at to this point and logarithms use a notation that will be new to almost everyone in an algebra class. However, you will find that once you get past the notation and start to understand some of their properties they really aren't too bad. So, we'll make sure to go over the notation and various properties of both exponential and logarithm functions in the hope that you will agree that once you understand those they aren't too bad.

In addition, we'll take a look at solving equations with exponentials and solving equations with logarithms. We'll also take a quick look a couple of applications involving exponentials and logarithms.

The following sections are the practice problems, with solutions, for this material.

6.1 Exponential Functions

1. Given the function $f(x) = 4^x$ evaluate each of the following.

(a) $f(-2)$ **(b)** $f\left(-\frac{1}{2}\right)$ **(c)** $f(0)$

(d) $f(1)$ **(e)** $f\left(\frac{3}{2}\right)$

Solutions

(a) $f(-2)$

Solution

All we need to do here is plug in the x and do any quick arithmetic we need to do.

$$f(-2) = 4^{-2} = \frac{1}{4^2} = \boxed{\frac{1}{16}}$$

(b) $f\left(-\frac{1}{2}\right)$

Solution

All we need to do here is plug in the x and do any quick arithmetic we need to do.

$$f\left(-\frac{1}{2}\right) = 4^{-\frac{1}{2}} = \frac{1}{4^{\frac{1}{2}}} = \frac{1}{\sqrt{4}} = \boxed{\frac{1}{2}}$$

(c) $f(0)$

Solution

All we need to do here is plug in the x and do any quick arithmetic we need to do.

$$f(0) = 4^0 = \boxed{1}$$

(d) $f(1)$

Solution

All we need to do here is plug in the x and do any quick arithmetic we need to do.

$$f(1) = 4^1 = \boxed{4}$$

(e) $f\left(\frac{3}{2}\right)$

Solution

All we need to do here is plug in the x and do any quick arithmetic we need to do.

$$f\left(\frac{3}{2}\right) = 4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = 2^3 = \boxed{8}$$

---

2. Given the function $f(x) = \left(\frac{1}{5}\right)^x$ evaluate each of the following.

(a) $f(-3)$ **(b)** $f(-1)$ **(c)** $f(0)$

(d) $f(2)$ **(e)** $f(3)$

Solutions

(a) $f(-3)$

Solution

All we need to do here is plug in the x and do any quick arithmetic we need to do.

$$f(-3) = \left(\frac{1}{5}\right)^{-3} = \left(\frac{5}{1}\right)^3 = \frac{5^3}{1^3} = \boxed{125}$$

(b) $f(-1)$

Solution

All we need to do here is plug in the x and do any quick arithmetic we need to do.

$$f(-) = \left(\frac{1}{5}\right)^{-1} = \left(\frac{5}{1}\right)^{1} = \boxed{5}$$

(c) $f(0)$

Solution

All we need to do here is plug in the x and do any quick arithmetic we need to do.

$$f(0) = \left(\frac{1}{5}\right)^{0} = \boxed{1}$$

(d) $f(2)$

Solution

All we need to do here is plug in the x and do any quick arithmetic we need to do.

$$f(2) = \left(\frac{1}{5}\right)^{2} = \frac{1^2}{5^2} = \boxed{\frac{1}{25}}$$

(e) $f(3)$

Solution

All we need to do here is plug in the x and do any quick arithmetic we need to do.

$$f(3) = \left(\frac{1}{5}\right)^{3} = \frac{1^3}{5^3} = \boxed{\frac{1}{125}}$$

---

3. Sketch each of the following.

(a) $f(x) = 6^x$ **(b)** $g(x) = 6^x - 9$ **(c)** $g(x) = 6^{x+1}$

Solutions

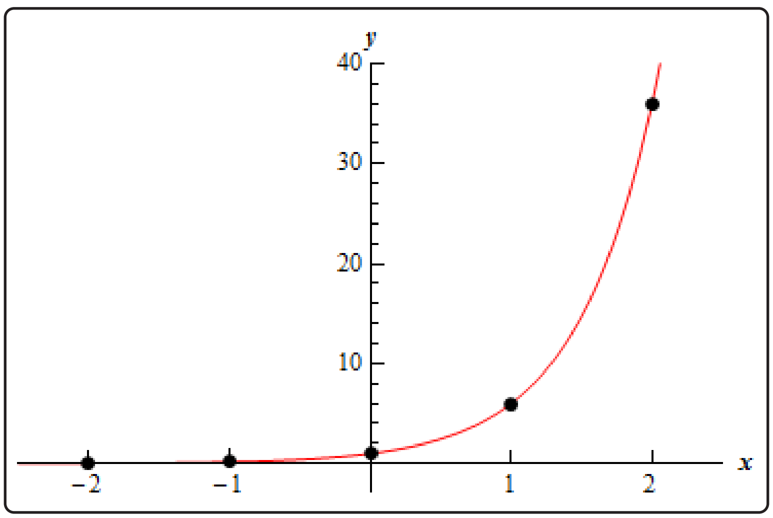
(a) $f(x) = 6^x$

Solution

We can build up a quick table of values that we can plot for the graph of this function.

x	$f(x)$
-2	$f(-2) = 6^{-2} = \frac{1}{6^2} = \frac{1}{36}$
-1	$f(-1) = 6^{-1} = \frac{1}{6}$
0	$f(0) = 6^0 = 1$
1	$f(1) = 6^1 = 6$
2	$f(2) = 6^2 = 36$

Here is a quick sketch of the graph of the function.



(b) $g(x) = 6^x - 9$

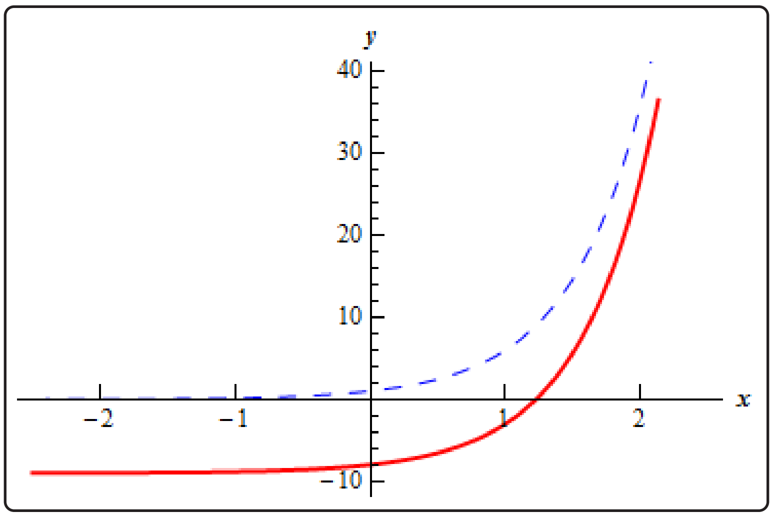
Solution

For this part all we need to do is recall the Transformations section from a couple of chapters ago. Using the "base" function of $f(x) = 6^x$ the function for this part can be written as,

$$g(x) = 6^x - 9 = f(x) - 9$$

Therefore, the graph for this part is just the graph of $f(x)$ shifted down by 9.

The graph of this function is shown below. The blue dashed line is the "base" function, $f(x)$, and the red solid line is the graph for this part, $g(x)$.



(c) $g(x) = 6^{x+1}$

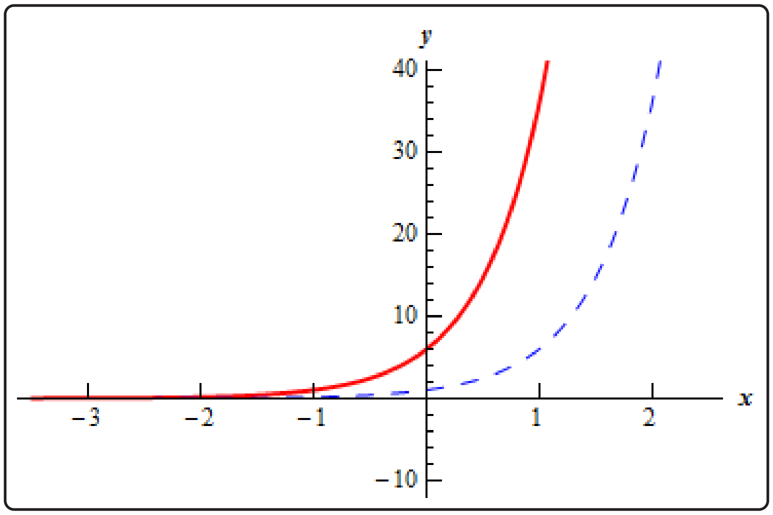
Solution

For this part all we need to do is recall the Transformations section from a couple of chapters ago. Using the "base" function of $f(x) = 6^x$ the function for this part can be written as,

$$g(x) = 6^{x+1} = f(x+1)$$

Therefore, the graph for this part is just the graph of $f(x)$ shifted left by 1.

The graph of this function is shown below. The blue dashed line is the "base" function, $f(x)$, and the red solid line is the graph for this part, $g(x)$.



4. Sketch the graph of $f(x) = \mathbf{e}^{-x}$.

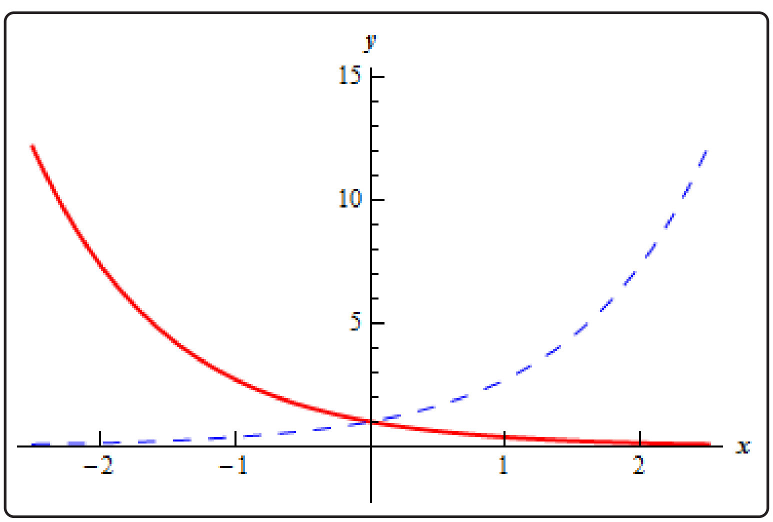
Solution

For this problem all we need to do is recall the Transformations section from a couple of chapters ago. Using the "base" function of $f(x) = \mathbf{e}^{x}$ the function for this part can be written as,

$$g(x) = \mathbf{e}^{-x} = f(-x)$$

Therefore, the graph for this part is just the graph of $f(x)$ reflected about the y-axis.

The graph of this function is shown below. The blue dashed line is the "base" function, $f(x)$, and the red solid line is the graph for this part, $g(x)$.



5. Sketch the graph of $f(x) = \mathbf{e}^{x-3} + 6$.

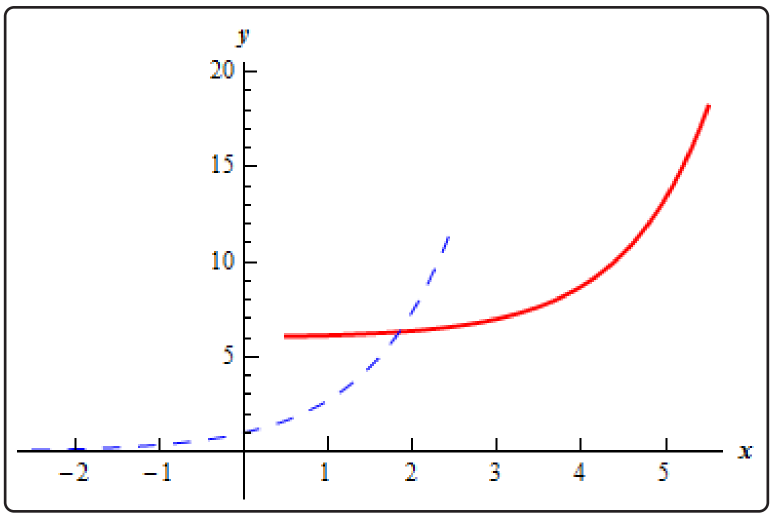
Solution

For this problem all we need to do is recall the Transformations section from a couple of chapters ago. Using the "base" function of $f(x) = \mathbf{e}^{x}$ the function for this part can be written as,

$$f(x) = \mathbf{e}^{x-3} + 6 = f(x-3) + 6$$

Therefore, the graph for this part is just the graph of $f(x)$ shifted right by 3 and up by 6.

The graph of this function is shown below. The blue dashed line is the "base" function, $f(x)$, and the red solid line is the graph for this part, $g(x)$.



6.2 Logarithm Functions

1. Write $7^5 = 16807$ in logarithmic form.

Solution

There really isn't all that much to do here other than refer to the definition of the logarithm function given in the notes for this section.

Here is the logarithmic form for this expression.

$$\boxed{\log_7 16807 = 5}$$

2. Write $16^{\frac{3}{4}} = 8$ in logarithmic form.

Solution

There really isn't all that much to do here other than refer to the definition of the logarithm function given in the notes for this section.

Here is the logarithmic form for this expression.

$$\boxed{\log_{16} 8 = \frac{3}{4}}$$

3. Write $\left(\frac{1}{3}\right)^{-2} = 9$ in logarithmic form.

Solution

There really isn't all that much to do here other than refer to the definition of the logarithm function given in the notes for this section.

Here is the logarithmic form for this expression.

$$\boxed{\log_{\frac{1}{3}} 9 = -2}$$

4. Write $\log_2 32 = 5$ in exponential form.

Solution

There really isn't all that much to do here other than refer to the definition of the logarithm function given in the notes for this section.

Here is the exponential form for this expression.

$$\boxed{2^5 = 32}$$

5. Write $\log_{\frac{1}{5}} \frac{1}{625} = 4$ in exponential form.

Solution

There really isn't all that much to do here other than refer to the definition of the logarithm function given in the notes for this section.

Here is the exponential form for this expression.

$$\boxed{\left(\frac{1}{5}\right)^4 = \frac{1}{625}}$$

6. Write $\log_9 \frac{1}{81} = -2$ in exponential form.

Solution

There really isn't all that much to do here other than refer to the definition of the logarithm function given in the notes for this section.

Here is the exponential form for this expression.

$$\boxed{9^{-2} = \frac{1}{81}}$$

7. Without using a calculator determine the exact value of $\log_3 81$.

Hint

Recall that converting a logarithm to exponential form can often help to evaluate these kinds of logarithms.

Step 1

Converting the logarithm to exponential form gives,

$$\log_3 81 = ? \qquad \Rightarrow \qquad 3^? = 81$$

Step 2

From this we can quickly see that $3^4 = 81$ and so we must have,

$$\boxed{\log_3 81 = 4}$$

8. Without using a calculator determine the exact value of $\log_5 125$.

Hint

Recall that converting a logarithm to exponential form can often help to evaluate these kinds of logarithms.

Step 1

Converting the logarithm to exponential form gives,

$$\log_5 125 = ? \qquad \Rightarrow \qquad 5^? = 125$$

Step 2

From this we can quickly see that $5^3 = 125$ and so we must have,

$$\boxed{\log_5 125 = 3}$$

9. Without using a calculator determine the exact value of $\log_2 \dfrac{1}{8}$.

Hint

Recall that converting a logarithm to exponential form can often help to evaluate these kinds of logarithms.

Step 1

Converting the logarithm to exponential form gives,

$$\log_2 \frac{1}{8} = ? \qquad \Rightarrow \qquad 2^{?} = \frac{1}{8}$$

Step 2

Now, we know that if we raise an integer to a negative exponent we'll get a fraction and so we must have a negative exponent and then we know that $2^3 = 8$. Therefore, we can see that $2^{-3} = \frac{1}{8}$ and so we must have,

$$\boxed{\log_2 \frac{1}{8} = -3}$$

---

10. Without using a calculator determine the exact value of $\log_{\frac{1}{4}} 16$.

Hint

Recall that converting a logarithm to exponential form can often help to evaluate these kinds of logarithms.

Step 1

Converting the logarithm to exponential form gives,

$$\log_{\frac{1}{4}} 16 = ? \qquad \Rightarrow \qquad \left(\frac{1}{4}\right)^{?} = 16$$

Step 2

Now, we know that if we raise an fraction to a power and get an integer out we must have had a negative exponent. Now, we also know that $4^2 = 16$. Therefore, we can see that $\left(\frac{1}{4}\right)^{-2} = \left(\frac{4}{1}\right)^{2} = 16$ and so we must have,

$$\boxed{\log_{\frac{1}{4}} 16 = -2}$$

---

11. Without using a calculator determine the exact value of $\ln \mathbf{e}^4$.

Hint

Recall that converting a logarithm to exponential form can often help to evaluate these kinds of logarithms. Also recall what the base is for a natural logarithm.

Step 1

Recalling that the base for a natural logarithm is **e** and converting the logarithm to exponential form gives,

$$\ln \mathbf{e}^4 = \log_{\mathbf{e}} \mathbf{e}^4 = ? \qquad \Rightarrow \qquad \mathbf{e}^? = \mathbf{e}^4$$

Step 2

From this we can quickly see that $\mathbf{e}^4 = \mathbf{e}^4$ and so we must have,

$$\boxed{\ln \mathbf{e}^4 = 4}$$

Note that an easier method of determining the value of this logarithm would have been to recall the properties of logarithm. In particular the property that states,

$$\log_b b^x = x$$

Using this we can also very quickly see what the value of the logarithm is.

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12. Without using a calculator determine the exact value of log $\frac{1}{100}$.

Hint

Recall that converting a logarithm to exponential form can often help to evaluate these kinds of logarithms. Also recall what the base is for a common logarithm.

Step 1

Recalling that the base for a common logarithm is 10 and converting the logarithm to exponential form gives,

$$\log\frac{1}{100} = \log_{10}\frac{1}{100} = ? \qquad \Rightarrow \qquad 10^{?} = \frac{1}{100}$$

Step 2

Now, we know that if we raise an integer to a negative exponent we'll get a fraction and so we must have a negative exponent and then we know that $10^2 = 100$. Therefore, we can see that $10^{-2} = \frac{1}{100}$ and so we must have,

$$\boxed{\log\frac{1}{100} = -2}$$

13. Write log $\left(3x^4y^{-7}\right)$ in terms of simpler logarithms.

Step 1

So, we're being asked here to use as many of the properties as we can to reduce this down into simpler logarithms.

First, we can use Property 5 to break up the product into individual logarithms. Note that just because the property only has two terms in it does not mean that it won't work for three (or more) terms. Here is the application of Property 5.

$$\log\left(3x^4y^{-7}\right) = \log(3) + \log\left(x^4\right) + \log\left(y^{-7}\right)$$

Step 2

Finally, we need to use Property 7 on the last two logarithms to bring the exponents out of the logarithms. Here is that work.

$$\log\left(3x^4y^{-7}\right) = \boxed{\log(3) + 4\log(x) - 7\log(y)}$$

Remember that we can only bring an exponent out of a logarithm if is on the whole argument of the logarithm. In other words, we couldn't bring any of the exponents out of the logarithms until we had dealt with the product.

---

14. Write $\ln\left(x\sqrt{y^2+z^2}\right)$ in terms of simpler logarithms.

Step 1

So, we're being asked here to use as many of the properties as we can to reduce this down into simpler logarithms.

First, we can use Property 5 to break up the product into individual logarithms. Here is that work.

$$\ln\left(x\sqrt{y^2+z^2}\right) = \ln(x) + \ln\left(\left(y^2+z^2\right)^{\frac{1}{2}}\right)$$

Note that we converted to root to a fractional exponent at the same time to help with the next step.

Step 2

Finally, we need to use Property 7 on the last logarithm to bring the root exponent out of the logarithm. Here is that work.

$$\ln\left(x\sqrt{y^2+z^2}\right) = \boxed{\ln(x) + \frac{1}{2}\ln\left(y^2+z^2\right)}$$

Remember that we can only bring an exponent out of a logarithm if is on the whole argument of the logarithm. In other words, we couldn't bring any of the exponents out of the logarithms until we had dealt with the product. Also, in the second logarithm while each term is squared the whole argument is not squared, *i.e.* it's not $(x+y)^2$ and so we can't bring those 2's out of the logarithm.

15. Write $\log_4\left(\dfrac{x-4}{y^2\,\sqrt[5]{z}}\right)$ in terms of simpler logarithms.

Step 1

So, we're being asked here to use as many of the properties as we can to reduce this down into simpler logarithms.

First, we can use Property 6 to break up the quotient into two logarithms. Here is that work.

$$\log_4\left(\frac{x-4}{y^2\,\sqrt[5]{z}}\right) = \log_4(x-4) - \log_4\left(y^2\,z^{\frac{1}{5}}\right)$$

Step 2

Next, we need to use Property 5 to break up the product in the second logarithm into two logarithms.

$$\log_4\left(\frac{x-4}{y^2\,\sqrt[5]{z}}\right) = \log_4(x-4) - \left(\log_4\left(y^2\right) + \log_4\left(z^{\frac{1}{5}}\right)\right)$$

Be careful with the minus sign that was in front of the second logarithm from Step 1! Because of that we need to have parenthesis on the product once we use Property 5. The sum of the two "smaller" logarithms is the same as the product logarithm from Step 1 and so because we have a minus sign in front of the product logarithm we also need to have a minus sign in front of the two logarithms after using Property 5. The only way to make sure of this is to use the parenthesis as shown.

Step 3

Finally, we'll distribute the minus sign through the parenthesis and then use Property 7 on the last two logarithms to bring the exponents out of the logarithms. Here is that work.

$$\log_4\left(\frac{x-4}{y^2\,\sqrt[5]{z}}\right) = \boxed{\log_4(x-4) - 2\log_4(y) - \frac{1}{5}\log_4(z)}$$

Remember that we can only bring an exponent out of a logarithm if is on the whole argument of the logarithm. In other words, we couldn't bring any of the exponents out of the logarithms until we had dealt with the quotient and product. Recall as well that we can't split up a sum/difference in a logarithm. Finally, make sure that you are careful in dealing with the minus sign we get from breaking up the quotient when dealing with the product in the denominator.

16. Combine $2\log_4 x + 5\log_4 y - \frac{1}{2}\log_4 z$ into a single logarithm with a coefficient of one.

Hint

The properties that we use to break up logarithms can be used in reverse as well.

Step 1

To convert this into a single logarithm we'll be using the properties that we used to break up logarithms in reverse. The first step in this process is to use the property,

$$\log_b\left(x^r\right) = r\log_b x$$

to make sure that all the logarithms have coefficients of one. This needs to be done first because all the properties that allow us to combine sums/differences of logarithms require coefficients of one on individual logarithms. So, using this property gives,

$$\log_4\left(x^2\right) + \log_4\left(y^5\right) - \log_4\left(z^{\frac{1}{2}}\right)$$

Step 2

Now, there are several ways to proceed from this point. We can use either of the two properties.

$$\log_b\left(xy\right) = \log_b x + \log_b y \qquad \log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

and in fact we'll need to use both in the end. We can use the product property on the first two logarithms (because they are a sum of logarithms) or the quotient property on the last two logarithms (because they are a difference of logarithms).

Which we use first does not matter as we'll end up with the same result in the end. For this problems we'll first use the product property on the first two logarithms to get,

$$2\log_4 x + 5\log_4 y - \frac{1}{2}\log_4 z = \log_4\left(x^2 y^5\right) - \log_4\left(\sqrt{z}\right)$$

Step 3

Finally, we can see that we have a difference of two logarithms left and so we'll use the quotient property to combine these to get,

$$2\log_4 x + 5\log_4 y - \frac{1}{2}\log_4 z = \boxed{\log_4\left(\frac{x^2y^5}{\sqrt{z}}\right)}$$

Note that the only reason we converted the fractional exponent to a root was to make the final answer a little nicer.

---

17. Combine $3\ln(t+5) - 4\ln(t) - 2\ln(s-1)$ into a single logarithm with a coefficient of one.

Hint

The properties that we use to break up logarithms can be used in reverse as well.

Step 1

To convert this into a single logarithm we'll be using the properties that we used to break up logarithms in reverse. The first step in this process is to use the property,

$$\log_b(x^r) = r\log_b x$$

to make sure that all the logarithms have coefficients of one. This needs to be done first because all the properties that allow us to combine sums/differences of logarithms require coefficients of one on individual logarithms. So, using this property gives,

$$\ln(t+5)^3 - \ln\left(t^4\right) - \ln(s-1)^2$$

Step 2

Now, there are several ways to proceed from this point. We can use either of the two properties.

$$\log_b(xy) = \log_b x + \log_b y \qquad \log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

and in fact we'll need to use both in the end.

We should also be careful with the fact that there are two minus signs in here as that sometimes adds confusion to the problem. They are easy to deal with however if we just factor a minus sign out of the last two terms to get,

$$3\ln(t+5) - 4\ln(t) - 2\ln(s-1) = \ln(t+5)^3 - \left(\ln\left(t^4\right) + \ln(s-1)^2\right)$$

Written in this form we can see that the last two logarithms are a sum and so we can use the product property to combine them to get,

$$3\ln(t+5) - 4\ln(t) - 2\ln(s-1) = \ln(t+5)^3 - \ln\left(t^4(s-1)^2\right)$$

Step 3

We now have a difference of two logarithms and we can use the quotient property to combine them to get,

$$3\ln(t+5) - 4\ln(t) - 2\ln(s-1) = \boxed{\ln\left[\frac{(t+5)^3}{t^4(s-1)^2}\right]}$$

18. Combine $\frac{1}{3}\log a - 6\log b + 2$ into a single logarithm with a coefficient of one.

Hint

The properties that we use to break up logarithms can be used in reverse as well. For the constant see if you figure out a way to write that as a logarithm.

Step 1

To convert this into a single logarithm we'll be using the properties that we used to break up logarithms in reverse. The first step in this process is to use the property,

$$\log_b(x^r) = r\log_b x$$

to make sure that all the logarithms have coefficients of one. This needs to be done first because all the properties that allow us to combine sums/differences of logarithms require coefficients of one on individual logarithms. So, using this property gives,

$$\log\left(a^{\frac{1}{3}}\right) - \log\left(b^6\right) + 2$$

Step 2

Now, for the 2 let's notice that we can write this in terms of a logarithm as,

$$2 = \log 10^2 = \log 100$$

Note that this is really just using the property,

$$\log_b b^x = x$$

So, we now have,

$$\log\left(a^{\frac{1}{3}}\right) - \log\left(b^6\right) + \log 100$$

Step 3

Now, there are several ways to proceed from this point. We can use either of the two properties.

$$\log_b(xy) = \log_b x + \log_b y \qquad \log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

and in fact we'll need to use both in the end.

The first two logarithms are a difference so let's use the quotient property to first combine those to get,

$$\log\left(a^{\frac{1}{3}}\right) - \log\left(b^6\right) + \log 10^2 = \log\left(\frac{\sqrt[3]{a}}{b^6}\right) + \log 100$$

We converted the fractional exponent in the first term to a root to make the answer a little nicer but doesn't really need to be done in general.

Step 4

Finally, note that we now have a sum of two logarithms and we can use the product property to combine those to get,

$$\log\left(a^{\frac{1}{3}}\right) - \log\left(b^6\right) + \log 100 = \boxed{\log\left(\frac{100\ \sqrt[3]{a}}{b^6}\right)}$$

19. Use the change of base formula and a calculator to find the value of $\log_{12} 35$.

Solution

We can use either the natural logarithm or the common logarithm to do this so we'll do both.

$$\log_{12} 35 = \frac{\ln 35}{\ln 12} = \frac{3.55534806}{2.48490665} = \boxed{1.43077731}$$

$$\log_{12} 35 = \frac{\log 35}{\log 12} = \frac{1.54406804}{1.07918125} = \boxed{1.43077731}$$

So, as we noted at the start it doesn't matter which logarithm we use we'll get the same answer in the end.

---

20. Use the change of base formula and a calculator to find the value of $\log_{\frac{2}{3}} 53$.

Solution

We can use either the natural logarithm or the common logarithm to do this so we'll do both.

$$\log_{\frac{2}{3}} 53 = \frac{\ln 53}{\ln \frac{2}{3}} = \frac{3.97029191}{-0.40546511} = \boxed{-9.79194469}$$

$$\log_{\frac{2}{3}} 53 = \frac{\log 53}{\log \frac{2}{3}} = \frac{1.72427587}{-0.17609126} = \boxed{-9.79194469}$$

So, as we noted at the start it doesn't matter which logarithm we use we'll get the same answer in the end.

---

21. Sketch the graph of $g(x) = -\ln(x)$.

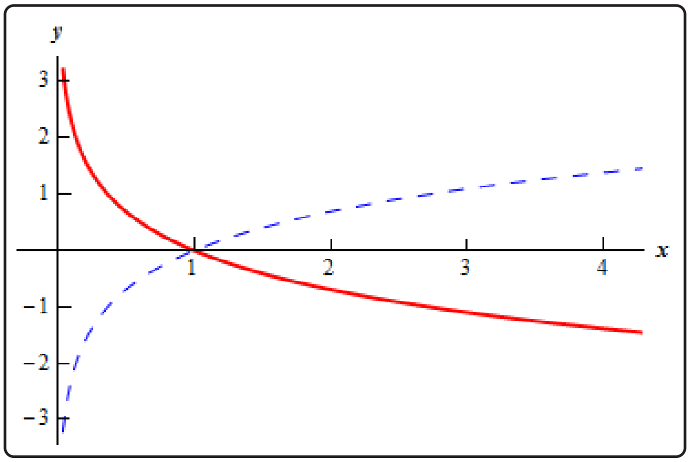
Solution

For this problem all we need to do is recall the Transformations section from a couple of chapters ago. Using the "base" function of $f(x) = \ln(x)$ the function for this part can be written as,

$$g(x) = -\ln(x) = -f(x)$$

Therefore, the graph for this part is just the graph of $f(x)$ reflected about the x-axis.

The graph of this function is shown below. The blue dashed line is the "base" function, $f(x)$, and the red solid line is the graph for this part, $g(x)$.



22. Sketch the graph of $g(x) = \ln(x + 5)$.

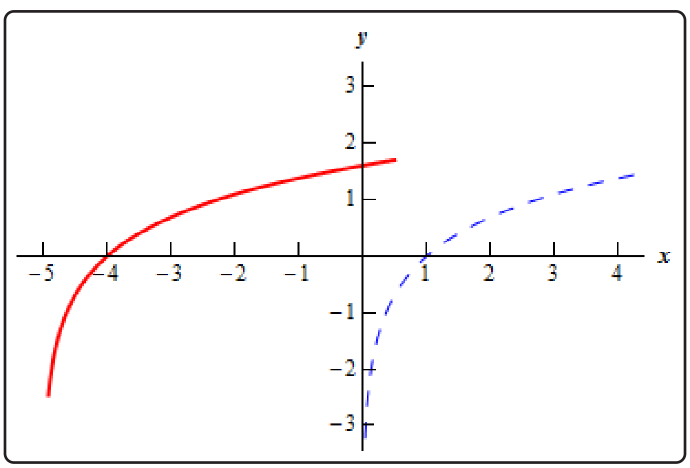
Solution

For this problem all we need to do is recall the Transformations section from a couple of chapters ago. Using the "base" function of $f(x) = \ln(x)$ the function for this part can be written as,

$$g(x) = \ln(x + 5) = f(x + 5)$$

Therefore, the graph for this part is just the graph of $f(x)$ shifted left by 5.

The graph of this function is shown below. The blue dashed line is the "base" function, $f(x)$, and the red solid line is the graph for this part, $g(x)$.



Do not get excited about the fact that we plugged negative values of x into the function! The problem with negative values is not the values we plug into a logarithm. Instead the problem with negative values is when we go to evaluate the logarithm.

It is perfectly fine to plug negative values into a logarithm as long as we don't end up actually evaluating a negative number. So, in this case we can see that as long as we require $x > -5$ then $x + 5 > 0$ and so those are acceptable values of x to plug in since we aren't going to evaluate negative number in the logarithm.

Note however that we do have avoid $x < -5$ since that would mean evaluating logarithms at negative numbers.

23. Sketch the graph of $g(x) = \ln(x) - 4$.

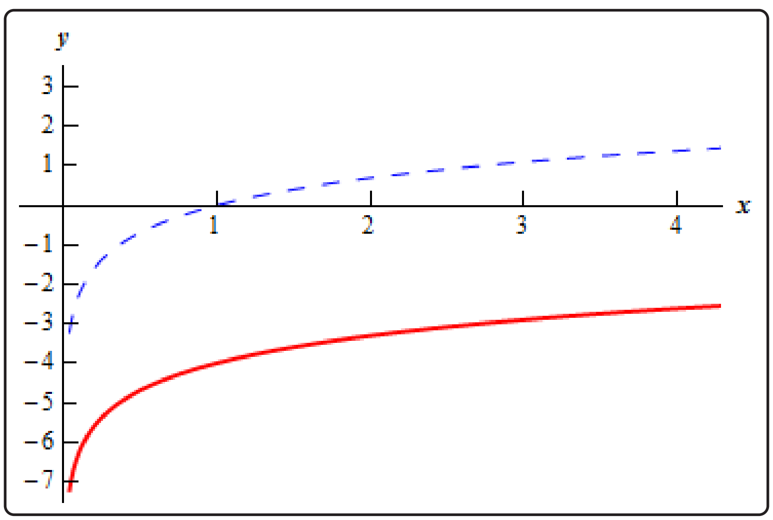
Solution

For this problem all we need to do is recall the Transformations section from a couple of chapters ago. Using the "base" function of $f(x) = \ln(x)$ the function for this part can be written as,

$$g(x) = \ln(x) - 4 = f(x) - 4$$

Therefore, the graph for this part is just the graph of $f(x)$ shifted down by 4.

The graph of this function is shown below. The blue dashed line is the "base" function, $f(x)$, and the red solid line is the graph for this part, $g(x)$.



6.3 Solving Exponential Functions

1. Solve the following equation.

$$6^{2x} = 6^{1-3x}$$

Step 1

Recall the property that says if $b^x = b^y$ then $x = y$. Since each exponential has the same base, 6 in this case, we can use this property to just set the exponents equal. Doing this gives,

$$2x = 1 - 3x$$

Step 2

Now all we need to do is solve the equation from Step 1 and that is a simple linear equation. Here is the solution work.

$$\begin{aligned} 2x &= 1 - 3x \\ 5x &= 1 \qquad \to \qquad x = \frac{1}{5} \end{aligned}$$

So, the solution to the equation is then : $\boxed{x = \frac{1}{5}}$.

---

2. Solve the following equation.

$$5^{1-x} = 25$$

Step 1

Recall the property that says if $b^x = b^y$ then $x = y$. In this case it looks like we can't use this property. However, recall that $25 = 5^2$ and if we write the right side of our equation using this we get,

$$5^{1-x} = 5^2$$

Now each exponential has the same base, 5 to be exact, so we can use this property to just set the exponents equal. Doing this gives,

$$1 - x = 2$$

Step 2

Now all we need to do is solve the equation from Step 1 and that is a simple linear equation. Here is the solution work.

$$\begin{aligned} 1 - x &= 2 \\ -x &= 1 \qquad \rightarrow \qquad x = -1 \end{aligned}$$

So, the solution to the equation is then : $\boxed{x = -1}$.

---

3. Solve the following equation.

$$8^{x^2} = 8^{3x+10}$$

Step 1

Recall the property that says if $b^x = b^y$ then $x = y$. Since each exponential has the same base, 8 in this case, we can use this property to just set the exponents equal. Doing this gives,

$$x^2 = 3x + 10$$

Step 2

Now all we need to do is solve the equation from Step 1 and that is a quadratic equation that we should be able to quickly solve. Here is the solution work.

$$\begin{aligned} x^2 &= 3x + 10 \\ x^2 - 3x - 10 &= 0 \\ (x-5)(x+2) &= 0 \qquad \rightarrow \qquad x = -2, \quad x = 5 \end{aligned}$$

So, the solutions to the equation are then : $\boxed{x = -2}$ and $\boxed{x = 5}$.

---

4. Solve the following equation.

$$7^{4-x} = 7^{4x}$$

Step 1

Recall the property that says if $b^x = b^y$ then $x = y$. Since each exponential has the same base, 7 in this case, we can use this property to just set the exponents equal. Doing this gives,

$$4 - x = 4x$$

Step 2

Now all we need to do is solve the equation from Step 1 and that is a simple linear equation. Here is the solution work.

$$\begin{aligned} 4 - x &= 4x \\ 4 &= 5x \qquad \rightarrow \qquad x = \frac{4}{5} \end{aligned}$$

So, the solution to the equation is then : $\boxed{x = \frac{4}{5}}$.

5. Solve the following equation.

$$2^{3x} = 10$$

Step 1

For this equation there is no way to easily get both sides with the same base. Therefore, we'll need to take the logarithm of both sides.

We can use any logarithm and the natural logarithm and common logarithm are usually good choices since most calculators can handle them. Because one of the bases in this equation is a 10 the common logarithm will probably be the better choice (although we can use the natural logarithm if we wanted to).

Taking the logarithm (using the common logarithm) of both sides gives,

$$\log 2^{3x} = \log 10$$

Step 2

Now we can easily compute the right side (which is also why we chose the common logarithm for this case) and we can use the logarithm property that says,

$$\log_b x^r = r\log_b x$$

to move the 3 x out of the exponent from the logarithm on the left. Doing this gives,

$$3x\left(\log 2\right) = x\left(3\log 2\right) = 1$$

We did a little rearranging of the left side to put all the numbers together in order to make the next step a little easier.

Step 3

Finally, all we need to do is solve for x. Recall that the equations at this step tend to look messier than we are used to dealing with. However, the logarithms in the equation at this point are just numbers and so we treat them as we treat all numbers with these kinds of equations.

In other words, all we need to do is divide both sides by the coefficient of the x and then user our calculators to get a decimal answer.

Here is the rest of the work for this problem.

$$x\left(3\log 2\right) = 1 \qquad \to \qquad x = \frac{1}{3\log 2} = \frac{1}{3\left(0.301029996\right)} = \boxed{1.10730936}$$

6. Solve the following equation.

$$7^{1-x} = 4^{3x+1}$$

Step 1

For this equation there is no way to easily get both sides with the same base. Therefore, we'll need to take the logarithm of both sides.

We can use any logarithm and the natural logarithm and common logarithm are usually good choices since most calculators can handle them. In this case there really isn't any reason to use one or the other so we'll use the natural logarithm (it's easier to write two letters - ln versus three letters - log after all...).

Taking the logarithm (using the natural logarithm) of both sides gives,

$$\ln 7^{1-x} = \ln 4^{3x+1}$$

Step 2

Now we can use the logarithm property that says,

$$\log_b x^r = r\log_b x$$

to move the exponents out of each of the logarithms. Doing this gives,

$$(1-x)\ln 7 = (3x+1)\ln 4$$

Step 3

Finally, all we need to do is solve for x. Recall that the equations at this step tend to look messier than we are used to dealing with. However, the logarithms in the equations at this point are just numbers and so we treat them as we treat all numbers with these kinds of equations. The work will be messier than we are used to but just keep in mind that the logarithms are just numbers!

Here is the rest of the work for this problem.

$$\begin{aligned}(1-x)\ln 7 &= (3x+1)\ln 4\\ \ln 7 - x\ln 7 &= 3x\ln 4 + \ln 4\\ \ln 7 - \ln 4 &= 3x\ln 4 + x\ln 7\\ \ln 7 - \ln 4 &= (3\ln 4 + \ln 7)\,x\\ x = \frac{\ln 7 - \ln 4}{3\ln 4 + \ln 7} &= \frac{1.945910149 - 1.386294361}{3\,(1.386294361) + 1.945910149} = \boxed{0.091668262}\end{aligned}$$

Again, the work is messier than we are used to but it is not really different from work we've done previously in solving equations. The answer is also going to be "messier" in the sense that it is a decimal and is liable to almost always be a decimal for most of these types of problems so don't worry about that.

7. Solve the following equation.

$$9 = 10^{4+6x}$$

Step 1

For this equation there is no way to easily get both sides with the same base. Therefore, we'll need to take the logarithm of both sides.

We can use any logarithm and the natural logarithm and common logarithm are usually good choices since most calculators can handle them. In this case one of the bases is a 10 and so the common logarithm is probably the better choice.

Taking the logarithm (using the common logarithm) of both sides gives,

$$\log 9 = \log 10^{4+6x}$$

Step 2

Now we can use the logarithm property that says,

$$\log 10^{f(x)} = f(x)$$

to simplify the right side of the equation. Doing this gives,

$$\log 9 = 4 + 6x$$

Step 3

Finally, all we need to do is solve for x. Recall that the equations at this step tend to look messier than we are used to dealing with. However, the logarithms in the equations at this point are just numbers and so we treat them as we treat all numbers with these kinds of equations. The work will be messier than we are used to but just keep in mind that the logarithms are just numbers!

Here is the rest of the work for this problem.

$$\begin{aligned} \log 9 &= 4 + 6x \\ \log 9 - 4 &= 6x \\ x &= \frac{\log 9 - 4}{6} = \frac{0.9542425094 - 4}{6} = \boxed{-0.5076262484} \end{aligned}$$

Again, the work is messier than we are used to but it is not really different from work we've done previously in solving equations. The answer is also going to be "messier" in the sense that it is a decimal and is liable to almost always be a decimal for most of these types of problems so don't worry about that.

Also, be careful when evaluating the numerator in the final answer. The 4 was outside of the logarithm and so cannot be moved into the logarithm. We probably should have been a little more careful with parenthesis and written the answer as,

$$x = \frac{\log(9) - 4}{6}$$

---

8. Solve the following equation.

$$\mathbf{e}^{7+2x} - 3 = 0$$

Step 1

Before we put any logarithms into this problem we first need to get the exponential on one side by itself so let's do that first.

$$\mathbf{e}^{7+2x} = 3$$

Step 2

Now we can take the logarithm of both sides and because we have a base of **e** in this problem the natural logarithm is probably the best choice. So, taking the logarithm (using the natural logarithm) of both sides gives,

$$\ln \mathbf{e}^{7+2x} = \ln 3$$

Step 3

Now we can use the logarithm property that says,

$$\ln \mathbf{e}^{f(x)} = f(x)$$

to simplify the left side of the equation. Doing this gives,

$$7 + 2x = \ln 3$$

Step 4

Finally, all we need to do is solve for x. Recall that the equations at this step tend to look messier than we are used to dealing with. However, the logarithms in the equations at this point are just numbers and so we treat them as we treat all numbers with these kinds of equations. The work will be messier than we are used to but just keep in mind that the logarithms are just numbers!

Here is the rest of the work for this problem.

$$\begin{aligned} 7 + 2x &= \ln 3 \\ 2x &= \ln 3 - 7 \\ x &= \frac{\ln 3 - 7}{2} = \frac{1.098612289 - 7}{2} = \boxed{-2.950693856} \end{aligned}$$

Again, the work is messier than we are used to but it is not really different from work we've done previously in solving equations. The answer is also going to be "messier" in the sense that it is a decimal and is liable to almost always be a decimal for most of these types of problems so don't worry about that.

Also, be careful when evaluating the numerator in the final answer. The 7 was outside of the logarithm and so cannot be moved into the logarithm. We probably should have been a little more careful with parenthesis and written the answer as,

$$x = \frac{\ln(3) - 7}{2}$$

which makes it a little clearer that the 7 isn't inside the logarithm. However, we typically don't put the parenthesis on the logarithm when it is just a number.

9. Solve the following equation.

$$\mathbf{e}^{4-7x} + 11 = 20$$

Step 1

Before we put any logarithms into this problem we first need to get the exponential on one side by itself so let's do that first.

$$\mathbf{e}^{4-7x} = 9$$

Step 2

Now we can take the logarithm of both sides and because we have a base of **e** in this problem the natural logarithm is probably the best choice. So, taking the logarithm (using the natural logarithm) of both sides gives,

$$\ln \mathbf{e}^{4-7x} = \ln 9$$

Step 3

Now we can use the logarithm property that says,

$$\ln \mathbf{e}^{f(x)} = f(x)$$

to simplify the left side of the equation. Doing this gives,

$$4 - 7x = \ln 9$$

Step 4

Finally, all we need to do is solve for x. Recall that the equations at this step tend to look messier than we are used to dealing with. However, the logarithms in the equations at this point are just numbers and so we treat them as we treat all numbers with these kinds of equations. The work will be messier than we are used to but just keep in mind that the logarithms are just numbers!

Here is the rest of the work for this problem.

$$\begin{aligned} 4 - 7x &= \ln 9 \\ -7x &= \ln 9 - 4 \\ x &= \frac{\ln 9 - 4}{-7} = \frac{2.197224577 - 4}{-7} = \boxed{0.2575393461} \end{aligned}$$

Again, the work is messier than we are used to but it is not really different from work we've done previously in solving equations. The answer is also going to be "messier" in the sense that it is a decimal and is liable to almost always be a decimal for most of these types of problems so don't worry about that.

Also, be careful when evaluating the numerator in the final answer. The 4 was outside of the logarithm and so cannot be moved into the logarithm. We probably should have been a little more careful with parenthesis and written the answer as,

$$x = \frac{\ln(9) - 4}{-7}$$

which makes it a little clearer that the 4 isn't inside the logarithm. However, we typically don't put the parenthesis on the logarithm when it is just a number.

6.4 Solving Logarithm Functions

1. Solve the following equation.

$$\log_4\left(x^2-2x\right)=\log_4\left(5x-12\right)$$

Hint

We had a very nice property from the notes on how to solve equations that contained exactly two logarithms with the same base! Also, don't forget that the values with get when we are done solving logarithm equations don't always correspond to actual solutions to the equation so be careful!

Step 1

Recall the property that says if $\log_b x = \log_b y$ then $x = y$. Since each logarithm is on opposite sides of the equal sign and each has the same base, 4 in this case, we can use this property to just set the arguments of each equal. Doing this gives,

$$x^2-2x=5x-12$$

Step 2

Now all we need to do is solve the equation from Step 1 and that is a quadratic equation that we know how to solve. Here is the solution work.

$$\begin{aligned} x^2-2x&=5x-12\\ x^2-7x+12&=0\\ (x-3)(x-4)&=0 \qquad \to \qquad x=3, \quad x=4 \end{aligned}$$

Step 3

As the final step we need to take each of the numbers from the above step and plug them into the original equation from the problem statement to make sure we don't end up taking the logarithm of zero or negative numbers!

Here is the checking work for each of the numbers.

$x=3$:

$$\begin{aligned} \log_4\left((3)^2-2(3)\right)&=\log_4\left(5(3)-12\right)\\ \log_4(3)&=\log_4(3) \qquad \text{OKAY} \end{aligned}$$

$x = 4$:

$$\log_4\left((4)^2 - 2(4)\right) = \log_4(5(4) - 12)$$
$$\log_4(8) = \log_4(8) \qquad \text{OKAY}$$

In this case, both numbers do not produce negative numbers in the logarithms and so they are in fact both solutions (won't happen with every problem so don't always expect this to happen!).

Therefore, the solutions to the equation are then : $\boxed{x = 3}$ and $\boxed{x = 4}$.

---

2. Solve the following equation.

$$\log(6x) - \log(4 - x) = \log(3)$$

Hint

We had a very nice property from the notes on how to solve equations that contained exactly two logarithms with the same base! Also, don't forget that the values with get when we are done solving logarithm equations don't always correspond to actual solutions to the equation so be careful!

Step 1

Recall the property that says if $\log_b x = \log_b y$ then $x = y$. That doesn't appear to have any use here since there are three logarithms in the equation. However, recall that we can combine a difference of logarithms (provide the coefficient of each is a one of course...) as follows,

$$\log\left(\frac{6x}{4 - x}\right) = \log(3)$$

We now have only two logarithms and each logarithm is on opposite sides of the equal sign and each has the same base, 10 in this case. Therefore, we can use this property to just set the arguments of each equal. Doing this gives,

$$\frac{6x}{4 - x} = 3$$

Step 2

Now all we need to do is solve the equation from Step 1 and that is an equation that we know how to solve. Here is the solution work.

$$\begin{aligned} \frac{6x}{4-x} &= 3 \\ 6x &= 3(4-x) = 12 - 3x \\ 9x &= 12 \qquad \to \qquad x = \frac{12}{9} = \frac{4}{3} \end{aligned}$$

Step 3

As the final step we need to take the number from the above step and plug it into the original equation from the problem statement to make sure we don't end up taking the logarithm of zero or negative numbers!

Here is the checking work for the number.

$x = \frac{4}{3}$:

$$\log\left(6\left(\frac{4}{3}\right)\right) - \log\left(4 - \frac{4}{3}\right) = \log(3)$$

$$\log(8) - \log\left(\frac{8}{3}\right) = \log(3) \qquad \text{OKAY}$$

In this case, the number did not produce negative numbers in the logarithms so it is in fact a solution (won't happen with every problem so don't always expect this to happen!).

Therefore, the solution to the equation is then : $\boxed{x = \frac{4}{3}}$.

3. Solve the following equation.

$$\ln(x) + \ln(x+3) = \ln(20-5x)$$

Hint

We had a very nice property from the notes on how to solve equations that contained exactly two logarithms with the same base! Also, don't forget that the values with get when we are done solving logarithm equations don't always correspond to actual solutions to the equation so be careful!

Step 1

Recall the property that says if $\log_b x = \log_b y$ then $x = y$. That doesn't appear to have any use here since there are three logarithms in the equation. However, recall that we can combine a sum of logarithms (provide the coefficient of each is a one of course...) as follows,

$$\ln(x(x+3)) = \ln(20-5x)$$

We now have only two logarithms and each logarithm is on opposite sides of the equal sign and each has the same base, **e** in this case. Therefore, we can use this property to just set the arguments of each equal. Doing this gives,

$$x(x+3) = 20-5x$$

Step 2

Now all we need to do is solve the equation from Step 1 and that is a quadratic equation that we know how to solve. Here is the solution work.

$$\begin{aligned} x(x+3) &= 20-5x \\ x^2+3x &= 20-5x \\ x^2+8x-20 &= 0 \\ (x+10)(x-2) &= 0 \qquad \to \qquad x=-10, \quad x=2 \end{aligned}$$

Step 3

As the final step we need to take each of the numbers from the above step and plug them into the original equation from the problem statement to make sure we don't end up taking the logarithm of zero or negative numbers!

Here is the checking work for each of the numbers.

$x = -10$:

$$\ln(-10) + \ln(-10+3) = \ln(20 - 5(-10))$$
$$\ln(-10) + \ln(-7) = \ln(70) \qquad \text{NOT OKAY}$$

$x = 2$:

$$\ln(2) + \ln(2+3) = \ln(20 - 5(2))$$
$$\ln(2) + \ln(5) = \ln(10) \qquad \text{OKAY}$$

In this case, the only one number did not produce negative numbers in the logarithms so that is the only number that will be a solution. The number that produced negative numbers in the logarithm is not a solution.

Therefore, the only solution to the equation is then : $\boxed{x = 2}$.

Note that it is vitally important that you do the check in the original equation. In the first step (where we combined two of the logarithms) we changed the equation and in the process introduced a number that is not in fact a solution.

Had we checked in any other equation in the solution work it would appear that $x = -10$ would be a solution to the equation. However, that is only because we were checking in a "modified" equation and not the original equation which is what we were being asked to solve.

This is always the danger of modifying equations during the solution process. Unfortunately, with many logarithm equations that is our only solution path and so is something that we need to be prepared to deal with.

4. Solve the following equation.

$$\log_3\left(25 - x^2\right) = 2$$

Hint

We had a very nice property from the notes on how to solve equations that contained exactly two logarithms with the same base and yes we can use that property here! Also, don't forget that the values with get when we are done solving logarithm equations don't always correspond to actual solutions to the equation so be careful!

Step 1

Recall the property that says if $\log_b x = \log_b y$ then $x = y$. That doesn't appear to have any use here since there is only one logarithm in the equation. Note however that we could write the right side, *i.e.* the 2, as,

$$2 = \log_3\left(3^2\right)$$

Doing this means we can write the equation as,

$$\log_3\left(25 - x^2\right) = \log_3\left(3^2\right) = \log_3\left(9\right)$$

We now have two logarithms and each logarithm is on opposite sides of the equal sign and each has the same base, 3 in this case. Therefore, we can use this property to just set the arguments of each equal. Doing this gives,

$$25 - x^2 = 9$$

Step 2

Now all we need to do is solve the equation from Step 1 and that is a quadratic equation that we know how to solve. Here is the solution work.

$$\begin{aligned} 25 - x^2 &= 9 \\ 16 &= x^2 \qquad \to \qquad x = \pm\sqrt{16} = \pm 4 \end{aligned}$$

Step 3

As the final step we need to take each of the numbers from the above step and plug them into the original equation from the problem statement to make sure we don't end up taking the logarithm of zero or negative numbers!

Here is the checking work for each of the numbers.

$x = -4$:

$$\begin{aligned} \log_3\left(25 - (-4)^2\right) &= 2 \\ \log_3\left(9\right) &= 2 \qquad \text{OKAY} \end{aligned}$$

$x = 4$:

$$\begin{aligned} \log_3\left(25 - (4)^2\right) &= 2 \\ \log_3\left(9\right) &= 2 \qquad \text{OKAY} \end{aligned}$$

In this case, both numbers do not produce negative numbers in the logarithms and so they are in fact both solutions (won't happen with every problem so don't always expect this to happen!).

Therefore, the solutions to the equation are then : $\boxed{x = -4}$ and $\boxed{x = 4}$.

Be careful to not make the mistake of assuming that just because a value of x is negative that it will automatically not be a solution to the equation. As we've shown here, even though $x = -4$ is negative it did not produce any negative values in the logarithms and so is perfectly acceptable as a solution.

5. Solve the following equation.

$$\log_2 (x+1) - \log_2 (2-x) = 3$$

Hint

If we can reduce all the logarithms to a single logarithm it would be quite easy to convert to exponential form. Also, don't forget that the values with get when we are done solving don't always correspond to actual solutions so be careful!

Step 1

First let's notice that we can combine the two logarithms on the left side to get,

$$\log_2 \left(\frac{x+1}{2-x}\right) = 3$$

Step 2

Now, we can easily convert this to exponential form.

$$\frac{x+1}{2-x} = 2^3 = 8$$

Step 3

Now all we need to do is solve the equation from Step 2 and that is an equation that we know how to solve. Here is the solution work.

$$\begin{aligned} \frac{x+1}{2-x} &= 8 \\ x+1 &= 8(2-x) = 16 - 8x \\ 9x = 15 \qquad &\to \qquad x = \frac{15}{9} = \frac{5}{3} \end{aligned}$$

Step 4

As the final step we need to take the number from the above step and plug it into the original equation from the problem statement to make sure we don't end up taking the logarithm of zero or negative numbers!

Here is the checking work for the number.

$x = \frac{5}{3}$:

$$\log_2\left(\frac{5}{3}+1\right) - \log_2\left(2-\frac{5}{3}\right) = 3$$
$$\log_2\left(\frac{8}{3}\right) - \log_2\left(\frac{1}{3}\right) = 3 \qquad \text{OKAY}$$

In this case, the number did not produce negative numbers in the logarithms so it is in fact a solution (won't happen with every problem so don't always expect this to happen!).

Therefore, the solution to the equation is then : $\boxed{x = \frac{5}{3}}$.

6. Solve the following equation.

$$\log_4(-x) + \log_4(6-x) = 2$$

Hint

If we can reduce all the logarithms to a single logarithm it would be quite easy to convert to exponential form. Also, don't forget that the values with get when we are done solving don't always correspond to actual solutions so be careful!

Step 1

First let's notice that we can combine the two logarithms on the left side to get,

$$\log_4(-x(6-x)) = 2$$

Step 2

Now, we can easily convert this to exponential form.

$$-x(6-x) = 4^2 = 16$$

Step 3

Now all we need to do is solve the equation from Step 2 and that is a quadratic equation that we know how to solve. Here is the solution work.

$$\begin{aligned} -x(6-x) &= 16 \\ x^2 - 6x - 16 &= 0 \\ (x-8)(x+2) &= 0 \qquad \to \qquad x = 8, \quad x = -2 \end{aligned}$$

Step 4

As the final step we need to take each of the numbers from the above step and plug them into the original equation from the problem statement to make sure we don't end up taking the logarithm of zero or negative numbers!

Here is the checking work for each of the numbers.

$x = 8$:

$$\begin{aligned} \log_4(-8) + \log_4(6-8) &= 2 \\ \log_4(-8) + \log_4(-2) &= 2 \qquad \text{NOT OKAY} \end{aligned}$$

$x = -2$:

$$\begin{aligned} \log_4(-(-2)) + \log_4(6-(-2)) &= 2 \\ \log_4(2) + \log_4(8) &= 2 \qquad \text{OKAY} \end{aligned}$$

In this case, the only one number did not produce negative numbers in the logarithms so that is the only number that will be a solution. The number that produced negative numbers in the logarithm is not a solution.

Therefore, the only solution to the equation is then : $\boxed{x = -2}$.

Be careful to not make the mistake of assuming that just because a value of x is negative that it will automatically not be a solution to the equation and just because a value of x is positive it will automatically be a solution. As we've shown here we can have negative values of x that are solutions and positive values of x that are not solutions.

Also note that it is vitally important that you do the check in the original equation. In the first step (where we combined two of the logarithms) we changed the equation and in the process introduced a number that is not in fact a solution.

Had we checked in any other equation in the solution work it would appear that $x = 8$ would be a solution to the equation. However, that is only because we were checking in a "modified" equation and not the original equation which is what we were being asked to solve.

This is always the danger of modifying equations during the solution process. Unfortunately, with many logarithm equations that is our only solution path and so is something that we need to be prepared to deal with.

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7. Solve the following equation.

$$\log\left(x\right) = 2 - \log\left(x - 21\right)$$

Hint

If we can reduce all the logarithms to a single logarithm it would be quite easy to convert to exponential form. Also, don't forget that the values with get when we are done solving don't always correspond to actual solutions so be careful!

Step 1

First let's notice that if we move the logarithm on the right side to the left side we can combine the two logarithms on the left side to get,

$$\begin{aligned}\log\left(x\right) &= 2 - \log\left(x - 21\right)\\ \log\left(x\right) + \log\left(x - 21\right) &= 2\\ \log\left(x\left(x - 21\right)\right) &= 2\end{aligned}$$

Step 2

Now, we can easily convert this to exponential form (recall that because there is no base given it is assumed to be 10!).

$$x\left(x - 21\right) = 10^2 = 100$$

Step 3

Now all we need to do is solve the equation from Step 2 and that is a quadratic equation that we know how to solve. Here is the solution work.

$$\begin{aligned}x\left(x - 21\right) &= 100\\ x^2 - 21x - 100 &= 0\\ \left(x - 25\right)\left(x + 4\right) &= 0 \qquad \rightarrow \qquad x = 25, \quad x = -4\end{aligned}$$

Step 4

As the final step we need to take each of the numbers from the above step and plug them into the original equation from the problem statement to make sure we don't end up taking the logarithm of zero or negative numbers!

Here is the checking work for each of the numbers.

$x = 25$:

$$\log(25) = 2 - \log(25 - 21)$$
$$\log(25) = 2 - \log(4) \qquad \text{OKAY}$$

$x = -4$:

$$\log(-4) = 2 - \log(-4 - 21)$$
$$\log(-4) = 2 - \log(-25) \qquad \text{NOT OKAY}$$

In this case, the only one number did not produce negative numbers in the logarithms so that is the only number that will be a solution. The number that produced negative numbers in the logarithm is not a solution.

Therefore, the only solution to the equation is then : $\boxed{x = 25}$.

Note that it is vitally important that you do the check in the original equation. In the first step (where we combined two of the logarithms) we changed the equation and in the process introduced a number that is not in fact a solution.

Had we checked in any other equation in the solution work it would appear that $x = -4$ would be a solution to the equation. However, that is only because we were checking in a "modified" equation and not the original equation which is what we were being asked to solve.

This is always the danger of modifying equations during the solution process. Unfortunately, with many logarithm equations that is our only solution path and so is something that we need to be prepared to deal with.

8. Solve the following equation.

$$\ln(x - 1) = 1 + \ln(3x + 2)$$

Hint

If we can reduce all the logarithms to a single logarithm it would be quite easy to convert to exponential form. Also, don't forget that the values with get when we are done solving don't always correspond to actual solutions so be careful!

Step 1

First let's notice that if we move the logarithm on the right side to the left side we can combine the two logarithms on the left side to get,

$$\begin{aligned}\ln(x-1) &= 1+\ln(3x+2)\\ \ln(x-1)-\ln(3x+2) &= 1\\ \ln\left(\frac{x-1}{3x+2}\right) &= 1\end{aligned}$$

Step 2

Now, we can easily convert this to exponential form (recall that because we are working with the natural logarithm the base is **e**!).

$$\frac{x-1}{3x+2} = \mathbf{e}^1 = \mathbf{e}$$

Step 3

Now all we need to do is solve the equation from Step 2 and that is an equation that we know how to solve. Here is the solution work.

$$\begin{aligned}\frac{x-1}{3x+2} &= \mathbf{e}\\ x-1 &= \mathbf{e}(3x+2)\\ x-1 &= 3\,\mathbf{e}\,x+2\mathbf{e}\\ x-3\,\mathbf{e}\,x &= 1+2\mathbf{e}\\ (1-3\mathbf{e})\,x &= 1+2\mathbf{e} \qquad \to \qquad x=\frac{1+2\mathbf{e}}{1-3\mathbf{e}} = -0.89961\end{aligned}$$

Do not get excited about the **e** in the equation. It works the same as if it was just a 4 or 5 or any other number. The only real difference is that the answer is a little messier that we usually get with these kinds of problems. Also, for the next step it is probably best to convert these kinds of numbers into decimal form.

Step 4

As the final step we need to take the number from the above step and plug it into the original equation from the problem statement to make sure we don't end up taking the logarithm of zero or negative numbers!

Here is the checking work for each of the numbers.

$x = -0.89961$:

$$\ln(-0.89961 - 1) = 1 + \ln(3(-0.89961) + 2)$$
$$\ln(-1.89961) = 1 + \ln(-0.69883) \qquad \text{NOT OKAY}$$

So, in this case the only number we got from Step 3 produced negative numbers in the logarithms and so can't be a solution. What this means for us is that there is **no solution** to this equation. This happens on occasion and we shouldn't worry about it when it does.

Note that it is vitally important that you do the check in the original equation. In the first step (where we combined two of the logarithms) we changed the equation and in the process introduced a number that is not in fact a solution.

Had we checked in any other equation in the solution work it would appear that $x = -0.89961$ would be a solution to the equation. However, that is only because we were checking in a "modified" equation and not the original equation which is what we were being asked to solve.

This is always the danger of modifying equations during the solution process. Unfortunately, with many logarithm equations that is our only solution path and so is something that we need to be prepared to deal with.

---

9. Solve the following equation.

$$2\log(x) - \log(7x - 1) = 0$$

Hint

If we can reduce all the logarithms to a single logarithm it would be quite easy to convert to exponential form. Also, don't forget that the values with get when we are done solving don't always correspond to actual solutions so be careful!

Step 1

First let's notice that we can move the 2 in front of the first logarithm into the logarithm as follows,

$$\log(x^2) - \log(7x - 1) = 0$$

We can now combine the two logarithms to get,

$$\log\left(\frac{x^2}{7x-1}\right)=0$$

Step 2

Now, we can easily convert this to exponential form (recall that because there is no base given it is assumed to be 10!).

$$\frac{x^2}{7x-1}=10^0=1$$

Step 3

Now all we need to do is solve the equation from Step 2 and that is a quadratic equation that we know how to solve. Here is the solution work.

$$\begin{aligned}\frac{x^2}{7x-1}&=1\\x^2&=7x-1\\x^2-7x+1&=0\end{aligned}$$

We can't factor this but we can use the quadratic formula on it. Doing that gives the following two numbers.

$$x=\frac{7\pm\sqrt{7^2-4(1)(1)}}{2(1)}=\frac{7\pm\sqrt{45}}{2}\qquad\rightarrow\qquad x=0.1459,\quad x=6.8541$$

Don't worry about the fact that we needed to use the quadratic formula to solve this. This will happen on occasion and we need to be able to deal with it when it happens.

Step 4

As the final step we need to take each of the numbers from the above step and plug them into the original equation from the problem statement to make sure we don't end up taking the logarithm of zero or negative numbers!

Here is the checking work for each of the numbers.

$x = 0.1459$:

$$\begin{aligned} 2\log(0.1459) - \log(7(0.1459) - 1) &= 0 \\ 2\log(0.1459) - \log(0.0213) &= 0 \qquad \text{OKAY} \end{aligned}$$

$x = 6.8541$:

$$\begin{aligned} 2\log(6.8541) - \log(7(6.8541) - 1) &= 0 \\ 2\log(6.8541) - \log(46.9787) &= 0 \qquad \text{OKAY} \end{aligned}$$

In this case, both numbers do not produce negative numbers in the logarithms and so they are in fact both solutions (won't happen with every problem so don't always expect this to happen!).

Therefore, the solutions to the equation are then : $\boxed{x = 0.1459}$ and $\boxed{x = 6.8541}$.

6.5 Applications

1. We have \$10,000 to invest for 44 months. How much money will we have if we put the money into an account that has an annual interest rate of 5.5% and interest is compounded

(a) quarterly **(b)** monthly **(c)** continuously

Solutions

(a) quarterly

Solution

From the problem statement we can see that,

$$P = 10000 \qquad r = \frac{5.5}{100} = 0.055 \qquad t = \frac{44}{12} = \frac{11}{3}$$

Remember that the value of r must be given as a decimal, *i.e.* the percentage divided by 100. Also remember that t must be in years and so we'll need to convert the months we are given to years.

For this part we are compounding interest rate quarterly and that means it will compound 4 times per year and so we also then know that,

$$m = 4$$

At this point all that we need to do is plug into the equation and run the numbers through a calculator to compute the amount of money that we'll have.

$$A = 10000\left(1 + \frac{0.055}{4}\right)^{\frac{11}{3}(4)} = 10000(1.01375)^{\frac{44}{3}} = 10000\,(1.221760422) = 12217.60$$

So, we'll have \$12,217.60 in the account after 44 months.

(b) monthly

Solution

From the problem statement we can see that,

$$P = 10000 \qquad r = \frac{5.5}{100} = 0.055 \qquad t = \frac{44}{12} = \frac{11}{3}$$

Remember that the value of r must be given as a decimal, *i.e.* the percentage divided by 100. Also remember that t must be in years and so we'll need to convert the months we are given to years.

For this part we are compounding interest rate monthly and that means it will compound 12 times per year and so we also then know that,

$$m = 12$$

At this point all that we need to do is plug into the equation and run the numbers through a calculator to compute the amount of money that we'll have.

$$A = 10000\left(1 + \frac{0.055}{12}\right)^{\frac{11}{3}(12)} = 10000(1.00453333)^{44} = 10000\,(1.222876562) = 12228.77$$

So, we'll have \$12,228.77 in the account after 44 months.

(c) continuously

Solution

From the problem statement we can see that,

$$P = 10000 \qquad r = \frac{5.5}{100} = 0.055 \qquad t = \frac{44}{12} = \frac{11}{3}$$

Remember that the value of r must be given as a decimal, *i.e.* the percentage divided by 100. Also remember that t must be in years and so we'll need to convert the months we are given to years.

For this part we are compounding continuously and so we won't have an m and will be using the other equation and all we have all we need to do the computation so,

$$A = 10000\mathbf{e}^{(0.055)\left(\frac{11}{3}\right)} = 10000\mathbf{e}^{0.2016666667} = 10000\,(1.223440127) = 12234.40$$

So, we'll have \$12,234.40 in the account after 44 months.

2. We are starting with \$5000 and we're going to put it into an account that earns an annual interest rate of 12%. How long should we leave the money in the account in order to double our money if interest is compounded

(a) quarterly **(b)** monthly **(c)** continuously

Solutions

(a) quarterly

Solution

From the problem statement we can see that,

$$A = 10000 \qquad P = 5000 \qquad r = \frac{12}{100} = 0.12$$

Remember that the value of r must be given as a decimal, *i.e.* the percentage divided by 100. Also, for this part we are compounding interest rate quarterly and that means it will compound 4 times per year and so we also then know that,

$$m = 4$$

Plugging into the equation gives us,

$$10000 = 5000\left(1 + \frac{0.12}{4}\right)^{4t} = 5000(1.03)^{4t}$$

Using the techniques from the Solve Exponential Equations section we can solve for t.

$$\begin{aligned} 2 &= 1.03^{4t} \\ \ln(2) &= \ln\left(1.03^{4t}\right) \\ \ln(2) &= 4t\ln(1.03) \\ t &= \boxed{\frac{\ln(2)}{4\ln(1.03)} = 5.8624} \end{aligned}$$

So, we'll double our money in approximately 5.8624 years.

(b) monthly

Solution

From the problem statement we can see that,

$$A = 10000 \qquad P = 5000 \qquad r = \frac{12}{100} = 0.12$$

Remember that the value of r must be given as a decimal, *i.e.* the percentage divided by 100. Also, for this part we are compounding interest rate monthly and that means it will compound 12 times per year and so we also then know that,

$$m = 12$$

Plugging into the equation gives us,

$$10000 = 5000\left(1 + \frac{0.12}{12}\right)^{12t} = 5000(1.01)^{12t}$$

Using the techniques from the Solve Exponential Equations section we can solve for t.

$$\begin{aligned} 2 &= 1.01^{12t} \\ \ln(2) &= \ln\left(1.01^{12t}\right) \\ \ln(2) &= 12t\ln(1.01) \\ t &= \boxed{\frac{\ln(2)}{12\ln(1.01)} = 5.8051} \end{aligned}$$

So, we'll double our money in approximately 5.8051 years.

(c) continuously

Solution

From the problem statement we can see that,

$$A = 10000 \qquad P = 5000 \qquad r = \frac{12}{100} = 0.12$$

Remember that the value of r must be given as a decimal, *i.e.* the percentage divided by 100. For this part we are compounding continuously and so we won't have an m and will be using the other equation.

Plugging into the continuously compounding interest equation gives,

$$10000 = 5000\mathbf{e}^{0.12t}$$

Now, solving this using the techniques from the Solve Exponential Equations section gives,

$$\begin{aligned} 2 &= \mathbf{e}^{0.12t} \\ \ln(2) &= \ln\left(\mathbf{e}^{0.12t}\right) \\ \ln(2) &= 0.12t \\ t &= \boxed{\frac{\ln(2)}{0.12} = 5.7762} \end{aligned}$$

So, we'll double our money in approximately 5.7762 years.

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3. A population of bacteria initially has 250 present and in 5 days there will be 1600 bacteria present.

(a) Determine the exponential growth equation for this population.

(b) How long will it take for the population to grow from its initial population of 250 to a population of 2000?

Solutions

(a) Determine the exponential growth equation for this population.

Solution

We can start off here by acknowledging that we know the initial number of bacteria is 250 and so $Q_0 = 250$. Therefore, the equation is then,

$$Q(t) = 250\mathbf{e}^{k\,t}$$

Now, we also know that $Q(5) = 1600$ and plugging this into the equation above gives,

$$1600 = Q(5) = 250\mathbf{e}^{5k}$$

We can use techniques from the Solve Logarithm Equations section to determine the value of k.

$$\begin{aligned} 1600 &= 250\mathbf{e}^{5k} \\ \frac{1600}{250} &= \mathbf{e}^{5k} \\ \ln\left(\frac{32}{5}\right) &= 5k \\ k &= \frac{1}{5}\ln\left(\frac{32}{5}\right) = 0.3712596 \end{aligned}$$

Depending upon your preferences we can use either the exact value or the decimal value. Note however that because k is in the exponent of an exponential function we'll need to use quite a few decimal places to avoid potentially large differences in the value that we'd get if we rounded off too much.

Putting all of this together the exponential growth equation for this population is,

$$\boxed{Q = 250\mathbf{e}^{\frac{1}{5}\ln\left(\frac{32}{5}\right)\, t}}$$

(b) How long will it take for the population to grow from its initial population of 250 to a population of 2000?

Solution

What we're really being asked to do here is to solve the equation,

$$2000 = Q(t) = 250\mathbf{e}^{\frac{1}{5}\ln\left(\frac{32}{5}\right)\, t}$$

and we know from the Solve Logarithm Equations section how to do that. Here is the solution work for this part.

$$\begin{aligned} \frac{2000}{250} &= \mathbf{e}^{\frac{1}{5}\ln\left(\frac{32}{5}\right)\, t} \\ \ln(8) &= \frac{1}{5}\ln\left(\frac{32}{5}\right)\, t \\ t &= \boxed{\frac{5\ln(8)}{\ln\left(\frac{32}{5}\right)} = 5.6010} \end{aligned}$$

It will take 5.601 days for the population to reach 2000.

4. We initially have 100 grams of a radioactive element and in 1250 years there will be 80 grams left.

(a) Determine the exponential decay equation for this element.

(b) How long will it take for half of the element to decay?

(c) How long will it take until there is only 1 gram of the element left?

Solutions

(a) Determine the exponential decay equation for this element.

Solution

We can start off here by acknowledging that we know the initial amount of the radioactive element is 100 and so $Q_0 = 100$. Therefore the equation is then,

$$Q(t) = 100\mathbf{e}^{k\,t}$$

Now, we also know that $Q(1250) = 80$ and plugging this into the equation above gives,

$$80 = Q(1250) = 100\mathbf{e}^{1250k}$$

We can use techniques from the Solve Logarithm Equations section to determine the value of k.

$$\begin{aligned} 80 &= 100\mathbf{e}^{1250k} \\ \frac{80}{100} &= \mathbf{e}^{1250k} \\ \ln\left(\frac{4}{5}\right) &= 1250k \\ k = \frac{1}{1250}\ln\left(\frac{4}{5}\right) &= -0.000178515 \end{aligned}$$

Depending upon your preferences we can use either the exact value or the decimal value. Note however that because k is in the exponent of an exponential function we'll need to use quite a few decimal places to avoid potentially large differences in the value that we'd get if we rounded off too much.

Putting all of this together the exponential decay equation for this population is,

$$\boxed{Q = 100\mathbf{e}^{\frac{1}{1250}\ln\left(\frac{4}{5}\right)\,t}}$$

(b) How long will it take for half of the element to decay?

Solution

What we're really being asked to do here is to solve the equation,

$$50 = Q(t) = 100\mathbf{e}^{\frac{1}{1250}\ln\left(\frac{4}{5}\right)t}$$

and we know from the Solve Logarithm Equations section how to do that. Here is the solution work for this part.

$$\begin{aligned} \frac{50}{100} &= \mathbf{e}^{\frac{1}{1250}\ln\left(\frac{4}{5}\right)t} \\ \ln\left(\frac{1}{2}\right) &= \frac{1}{1250}\ln\left(\frac{4}{5}\right)t \\ t &= \boxed{\frac{1250\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{4}{5}\right)} = 3882.8546} \end{aligned}$$

It will take 3882.8546 years for half of the element to decay. On a side note this time is called the **half-life** of the element.

(c) How long will it take until there is only 1 gram of the element left?

Solution

In this part we're being asked to solve the equation,

$$1 = Q(t) = 100\mathbf{e}^{\frac{1}{1250}\ln\left(\frac{4}{5}\right)t}$$

The solution process for this part is the same as that for the previous part. Here is the solution work for this part.

$$\begin{aligned} \frac{1}{100} &= \mathbf{e}^{\frac{1}{1250}\ln\left(\frac{4}{5}\right)t} \\ \ln\left(\frac{1}{100}\right) &= \frac{1}{1250}\ln\left(\frac{4}{5}\right)t \\ t &= \boxed{\frac{1250\ln\left(\frac{1}{100}\right)}{\ln\left(\frac{4}{5}\right)} = 25797.1279} \end{aligned}$$

There will only be 1 gram of the element left after 25,797.1279 years.

7 Systems of Equations

This is a fairly short chapter devoted to solving systems of equations. A system of equations is a set of equations each containing one or more variable.

We will focus exclusively on systems of two equations with two unknowns and three equations with three unknowns although the methods looked at here can be easily extended to more equations. We'll look at a couple of methods of solving systems including the idea of an augmented matrix to solve systems.

In addition, with the exception of the last section we will be dealing only with systems of linear equations.

The following sections are the practice problems, with solutions, for this material.

7.1 Linear Systems of Two Variables

1. Use the Method of Substitution to find the solution to the following system or to determine if the system is inconsistent or dependent.

$$\begin{aligned} x - 7y &= -11 \\ 5x + 2y &= -18 \end{aligned}$$

Step 1

Before we get started with the solution process for this system we need to make it clear that there is no "one correct solution path". There are lots of solution paths that we can take to find the solution to this system. All are correct and all will end up with the same solution to the system (provided the work has been done correctly of course...).

Okay, let's get started on the solution to this system.

The Method of Substitution tells us that we first need to solve one of the equations for one of the variables. The equation we solve and the variable we solve for technically doesn't matter as noted above.

However, there is often one equation/variable combination that is "easier" than the others. In this case we can quickly solve the first equation for x without a lot of extra work so let's do that.

$$x - 7y = -11 \qquad \Rightarrow \qquad x = 7y - 11$$

Step 2

We now take the equation for x we found above and substitute this into the other equation (the second equation in this case). Doing this gives,

$$\begin{aligned} 5x + 2y &= -18 \\ 5\left(7y - 11\right) + 2y &= -18 \end{aligned}$$

Step 3

We can now solve the equation we found in the previous step for y. Doing this gives,

$$\begin{aligned} 5\left(7y - 11\right) + 2y &= -18 \\ 35y - 55 + 2y &= -18 \\ 37y &= 37 \qquad \to \qquad y = 1 \end{aligned}$$

Step 4

Finally, we can plug the value of y we found in the previous step into the equation for x we found in the first step. This gives,

$$x = 7(1) - 11 = -4$$

The solution to the system is then : $\boxed{x = -4,\ y = 1}$.

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2. Use the Method of Substitution to find the solution to the following system or to determine if the system is inconsistent or dependent.

$$\begin{aligned} 7x - 8y &= -12 \\ -4x + 2y &= 3 \end{aligned}$$

Step 1

Before we get started with the solution process for this system we need to make it clear that there is no "one correct solution path". There are lots of solution paths that we can take to find the solution to this system. All are correct and all will end up with the same solution to the system (provided the work has been done correctly of course...).

Okay, let's get started on the solution to this system.

The Method of Substitution tells us that we first need to solve one of the equations for one of the variables. The equation we solve and the variable we solve for technically doesn't matter as noted above.

However, there is often one equation/variable combination that is "easier" than the others. In this case we can solve the second equation for y without a lot of extra work so let's do that.

$$\begin{aligned} -4x + 2y &= 3 \\ 2y &= 4x + 3 \qquad \Rightarrow \qquad y = 2x + \frac{3}{2} \end{aligned}$$

Note that you will often get fractions showing up at this step and there isn't going to be a whole lot that you can do about it so don't worry when they show up!

Step 2

We now take the equation for y we found above and substitute this into the other equation (the first equation in this case). Doing this gives,

$$\begin{aligned} 7x - 8y &= -12 \\ 7x - 8\left(2x + \frac{3}{2}\right) &= -12 \end{aligned}$$

Step 3

We can now solve the equation we found in the previous step for x. Doing this gives,

$$\begin{aligned} 7x - 8\left(2x + \frac{3}{2}\right) &= -12 \\ 7x - 16x - 12 &= -12 \\ -9x &= 0 \qquad \to \qquad x = 0 \end{aligned}$$

Do not get excited about the zero here! They will be answers occasionally.

Step 4

Finally, we can plug the value of x we found in the previous step into the equation for y we found in the first step. This gives,

$$y = 2\left(0\right) + \frac{3}{2} = \frac{3}{2}$$

The solution to the system is then : $\boxed{x = 0,\ y = \frac{3}{2}}$.

3. Use the Method of Substitution to find the solution to the following system or to determine if the system is inconsistent or dependent.

$$\begin{aligned} 3x + 9y &= -6 \\ -4x - 12y &= 8 \end{aligned}$$

Step 1

Before we get started with the solution process for this system we need to make it clear that there is no "one correct solution path". There are lots of solution paths that we can take to find the solution to this system. All are correct and all will end up with the same solution to the system (provided the work has been done correctly of course...).

Okay, let's get started on the solution to this system.

The Method of Substitution tells us that we first need to solve one of the equations for one of the variables. The equation we solve and the variable we solve for technically doesn't matter as noted above.

In this case both equations seem equally "easy" to deal with and so let's solve the second equation for x since that is a combination we didn't use in the first couple of problems.

$$\begin{aligned} -4x - 12y &= 8 \\ -4x &= 12y + 8 \qquad \Rightarrow \qquad x = -3y - 2 \end{aligned}$$

Step 2

We now take the equation for x we found above and substitute this into the other equation (the first equation in this case). Doing this gives,

$$\begin{aligned} 3x + 9y &= -6 \\ 3\left(-3y - 2\right) + 9y &= -6 \end{aligned}$$

Step 3

We can now solve the equation we found in the previous step for y. Doing this gives,

$$\begin{aligned} 3\left(-3y - 2\right) + 9y &= -6 \\ -9y - 6 + 9y &= -6 \\ -6 &= -6 \end{aligned}$$

Step 4

Now, the result from the previous step is true for any value of y or x and so we know that the system is **dependent** and there will be an infinite number of solutions to the system. We can write the “solution” to this system as follows,

$$\begin{aligned} x &= -3t - 2 \\ y &= t \end{aligned} \qquad t \text{ is any number}$$

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4. Use the Method of Elimination to find the solution to the following system or to determine if the system is inconsistent or dependent.

$$\begin{aligned} 6x - 5y &= 8 \\ -12x + 2y &= 0 \end{aligned}$$

Step 1

Before we get started with the solution process for this system we need to make it clear that there is no “one correct solution path”. There are lots of solution paths that we can take to find the solution to this system. All are correct and all will end up with the same solution to the system (provided the work has been done correctly of course...).

Okay, let’s get started on the solution to this system.

The Method of Elimination tells us that we first need to multiply one or both of the equations by constants so that one of the variables has the same coefficient but with opposite signs and then add the two equations.

For this system if we multiply the first equation by 2 then the first equation will have an x coefficient of 12 while the second equation will have an x coefficient of -12. This is exactly what we need so we'll do that and then add the resulting equations.

$$\begin{array}{rcr} 6x - 5y = 8 & \xrightarrow{\times 2} & 12x - 10y = 16 \\ -12x + 2y = 0 & \xrightarrow{\text{same}} & \underline{-12x + 2y = 0} \\ & & -8y = 16 \end{array}$$

Step 2

We can now easily solve the result from the above step to see that $y = -2$.

Step 3

Finally, we can plug the value of y we found in the previous step in either of the original equations and solve for x. We'll use the first equation for this.

$$\begin{aligned} 6x - 5\left(-2\right) &= 8 \\ 6x + 10 &= 8 \\ 6x &= -2 \qquad \to \qquad x = -\frac{1}{3} \end{aligned}$$

The solution to the system is then : $\boxed{x = -\frac{1}{3},\ y = -2}$.

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5. Use the Method of Elimination to find the solution to the following system or to determine if the system is inconsistent or dependent.

$$\begin{aligned} -2x + 10y &= 2 \\ 5x - 25y &= 3 \end{aligned}$$

Step 1

Before we get started with the solution process for this system we need to make it clear that there is no "one correct solution path". There are lots of solution paths that we can take to find the solution to this system. All are correct and all will end up with the same solution to the system (provided the work has been done correctly of course...).

Okay, let's get started on the solution to this system.

The Method of Elimination tells us that we first need to multiply one or both of the equations by constants so that one of the variables has the same coefficient but with opposite signs and then add the two equations.

For this system if we multiply the first equation by 5 and the second equation by 2 then the first equation will have an x coefficient of -10 while the second equation will have an x coefficient of 10. This is exactly what we need so we'll do that and then add the resulting equations.

$$\begin{aligned} -2x + 10y &= 2 & \xrightarrow{\times 5} \qquad & -10x + 50y = 10 \\ 5x - 25y &= 3 & \xrightarrow{\times 2} \qquad & \underline{10x - 50y = 6} \\ & & & \qquad\qquad 0 = 16 \end{aligned}$$

Step 2

The result above is clearly not true and so this system is **inconsistent** and has **no solution**.

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6. Use the Method of Elimination to find the solution to the following system or to determine if the system is inconsistent or dependent.

$$\begin{aligned} 2x + 3y &= 20 \\ 7x + 2y &= 53 \end{aligned}$$

Step 1

Before we get started with the solution process for this system we need to make it clear that there is no "one correct solution path". There are lots of solution paths that we can take to find the solution to this system. All are correct and all will end up with the same solution to the system (provided the work has been done correctly of course...).

Okay, let's get started on the solution to this system.

The Method of Elimination tells us that we first need to multiply one or both of the equations by constants so that one of the variables has the same coefficient but with opposite signs and then add the two equations.

For this system if we multiply the first equation by 2 and the second equation by -3 then the first equation will have a y coefficient of 6 while the second equation will have a y coefficient of -6. This is exactly what we need so we'll do that and then add the resulting equations.

$$\begin{array}{rcr} 2x + 3y = 20 & \xrightarrow{\times\ 2} & 4x + 6y = 40 \\ 7x + 2y = 53 & \xrightarrow{\times\ -3} & \underline{-21x - 6y = -159} \\ & & -17x = -119 \end{array}$$

Step 2

We can now easily solve the result from the above step to see that $x = 7$.

Step 3

Finally, we can plug the value of y we found in the previous step in either of the original equations and solve for y. We'll use the first equation for this.

$$\begin{aligned} 7\left(7\right)+2y&=53 \\ 49+2y&=53 \\ 2y&=4 \qquad \to \qquad y=2 \end{aligned}$$

The solution to the system is then : $\boxed{x=7,\ y=2}$.

7.2 Linear Systems of Three Variables

1. Find the solution to the following system of equations.

$$\begin{aligned} 2x + 5y + 2z &= -38 \\ 3x - 2y + 4z &= 17 \\ -6x + y - 7z &= -12 \end{aligned}$$

Step 1

Before we get started with the solution process for this system we need to make it clear that there is no "one correct solution path". There are lots of solution paths that we can take to find the solution to this system. All are correct and all will end up with the same solution to the system (provided the work has been done correctly of course...).

Okay, let's get started on the solution to this system.

For this system it looks like if we multiply the first equation by 3 and the second equation by 2 both of these equations will have x coefficients of 6 which we can then eliminate if we add the third equation to each of them.

So, let's first do the multiplication.

$$\begin{aligned} 2x + 5y + 2z &= -38 & \xrightarrow{\times 3} && 6x + 15y + 6z &= -114 \\ 3x - 2y + 4z &= 17 & \xrightarrow{\times 2} && 6x - 4y + 8z &= 34 \\ -6x + y - 7z &= -12 & \xrightarrow{\text{same}} && -6x + y - 7z &= -12 \end{aligned}$$

Step 2

Okay, we'll now replace the first equation with the sum of the first and third equation and we'll replace the second equation with the sum of the second and third equation. Here is the result from doing those operations.

$$\begin{aligned} 16y - z &= -126 \\ -3y + z &= 22 \\ -6x + y - 7z &= -12 \end{aligned}$$

Step 3

Next notice that we can eliminate z from the first equation simply by replacing it with the sum of the first and second equation. Here is the result from that operation.

$$\begin{aligned} 13y \qquad &= -104 \\ -3y + z &= 22 \\ -6x + y - 7z &= -12 \end{aligned}$$

Step 4

Okay, from the first equation we can see that we must have $y = -8$.

Step 5

We can plug $y = -8$ into the second equation and solve that for z.

$$-3\left(-8\right) + z = 22 \qquad \rightarrow \qquad z = -2$$

Step 6

Finally, plug $y = -8$ and $z = -2$ into the third equation and solve for x.

$$\begin{aligned} -6x + \left(-8\right) - 7\left(-2\right) &= -12 \\ -6x + 6 &= -12 \qquad \rightarrow \qquad x = 3 \end{aligned}$$

The solution to the system is then : $\boxed{x = 3,\ y = -8,\ z = -2}$.

2. Find the solution to the following system of equations.

$$\begin{aligned} 3x - 9z &= 33 \\ 7x - 4y - z &= -15 \\ 4x + 6y + 5z &= -6 \end{aligned}$$

Step 1

Before we get started with the solution process for this system we need to make it clear that there is no "one correct solution path". There are lots of solution paths that we can take to find the solution to this system. All are correct and all will end up with the same solution to the system (provided the work has been done correctly of course...).

Okay, let's get started on the solution to this system.

For this system it looks like we can easily solve the first equation for x and get an equation involving only z which we can in turn plug in the second and third equation.

Here is the first equation solved for x.

$$\begin{aligned} 3x - 9z &= 33 \\ 3x &= 9z + 33 \qquad \to \qquad x = 3z + 11 \end{aligned}$$

Step 2

Plugging the equation we found above into the second and third equations and doing some simplification gives,

$$\begin{aligned} 7(3z+11) - 4y - z &= -15 \\ 4(3z+11) + 6y + 5z &= -6 \end{aligned} \qquad \to \qquad \begin{aligned} -4y + 20z &= -92 \\ 6y + 17z &= -50 \end{aligned}$$

Step 3

Now, notice that if we multiply the first equation above by 3 and the second equation above by 2 we can cancel the y's when we add the results. Here is that work.

$$\begin{aligned} -4y + 20z &= -92 & \xrightarrow{\times\, 3} \qquad -12y + 60z &= -276 \\ 6y + 17z &= -50 & \xrightarrow{\times\, 2} \qquad \underline{12y + 34z} &\underline{= -100} \\ & & 94z &= -376 \end{aligned}$$

Step 4

From the equation above we can see that we must have $z = -4$.

Step 5

We can plug $z = -4$ into either of the equations we got in Step 2 and solve for y. We'll use the second equation for this propose.

$$6y + 17\left(-4\right) = -50 \qquad \to \qquad y = 3$$

Step 6

Finally, plug $z = -4$ we got in Step 1 to determine the value of x.

$$x = 3\left(-4\right) + 11 = -1$$

The solution to the system is then : $\boxed{x = -1,\ y = 3,\ z = -4}$.

7.3 Augmented Matrix

1. For the following augmented matrix perform the indicated elementary row operations.

$$\left[\begin{array}{rrr|r} 4 & -1 & 3 & 5 \\ 0 & 2 & 5 & 9 \\ -6 & 1 & -3 & 10 \end{array}\right]$$

(a) $8R_1$ **(b)** $R_2 \leftrightarrow R_3$ **(c)** $R_2 + 3R_1 \rightarrow R_2$

Solutions

(a) $8R_1$

Solution

This operation is telling us to multiply all the entries in Row 1 of the augmented matrix by 8 so let's do that.

$$\left[\begin{array}{rrr|r} 4 & -1 & 3 & 5 \\ 0 & 2 & 5 & 9 \\ -6 & 1 & -3 & 10 \end{array}\right] \xrightarrow{8R_1} \left[\begin{array}{rrr|r} 32 & -8 & 24 & 40 \\ 0 & 2 & 5 & 9 \\ -6 & 1 & -3 & 10 \end{array}\right]$$

(b) $R_2 \leftrightarrow R_3$

Solution

This operation is telling us to interchange Row 2 and Row 3 of the augmented matrix. Here is that work.

$$\left[\begin{array}{rrr|r} 4 & -1 & 3 & 5 \\ 0 & 2 & 5 & 9 \\ -6 & 1 & -3 & 10 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{rrr|r} 4 & -1 & 3 & 5 \\ -6 & 1 & -3 & 10 \\ 0 & 2 & 5 & 9 \end{array}\right]$$

(c) $R_2 + 3R_1 \to R_2$

Solution

For this operation we are going to replace Row 2 with the results of taking the original entries from Row 2 and add to them 3 times the entries in Row 1.

$$\left[\begin{array}{rrr|r} 4 & -1 & 3 & 5 \\ 0 & 2 & 5 & 9 \\ -6 & 1 & -3 & 10 \end{array}\right] \begin{array}{c} R_2 + 3R_1 \to R_2 \\ \to \end{array} \left[\begin{array}{rrr|r} 4 & -1 & 3 & 5 \\ 12 & -1 & 14 & 24 \\ -6 & 1 & -3 & 10 \end{array}\right]$$

Here are the individual computations for this operation.

$$\begin{aligned} \text{Column 1} &: \ 0 + 3(4) = 12 \\ \text{Column 2} &: \ 2 + 3(-1) = -1 \\ \text{Column 3} &: \ 5 + 3(3) = 14 \\ \text{Column 4} &: \ 9 + 3(5) = 24 \end{aligned}$$

2. For the following augmented matrix perform the indicated elementary row operations.

$$\left[\begin{array}{rrr|r} 1 & -6 & 2 & 0 \\ 2 & -8 & 10 & 4 \\ 3 & -4 & -1 & 2 \end{array}\right]$$

(a) $\frac{1}{2}R_2$ **(b)** $R_1 \leftrightarrow R_3$ **(c)** $R_1 - 6R_3 \to R_1$

Solutions

(a) $\frac{1}{2}R_2$

Solution

This operation is telling us to multiply all the entries in Row 2 of the augmented matrix by ½ so let's do that.

$$\left[\begin{array}{rrr|r} 1 & -6 & 2 & 0 \\ 2 & -8 & 10 & 4 \\ 3 & -4 & -1 & 2 \end{array}\right] \begin{array}{c} \frac{1}{2}R_2 \\ \to \end{array} \left[\begin{array}{rrr|r} 1 & -6 & 2 & 0 \\ 1 & -4 & 5 & 2 \\ 3 & -4 & -1 & 2 \end{array}\right]$$

(b) $R_1 \leftrightarrow R_3$

Solution

This operation is telling us to interchange Row 1 and Row 3 of the augmented matrix. Here is that work.

$$\left[\begin{array}{rrr|r} 1 & -6 & 2 & 0 \\ 2 & -8 & 10 & 4 \\ 3 & -4 & -1 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{rrr|r} 3 & -4 & -1 & 2 \\ 2 & -8 & 10 & 4 \\ 1 & -6 & 2 & 0 \end{array}\right]$$

(c) $R_1 - 6R_3 \rightarrow R_1$

Solution

For this operation we are going to replace Row 1 with the results of taking the original entries from Row 1 and subtract from them 6 times the entries in Row 3.

$$\left[\begin{array}{rrr|r} 1 & -6 & 2 & 0 \\ 2 & -8 & 10 & 4 \\ 3 & -4 & -1 & 2 \end{array}\right] \xrightarrow{R_1 - 6R_3 \rightarrow R_1} \left[\begin{array}{rrr|r} -17 & 18 & 8 & -12 \\ 2 & -8 & 10 & 4 \\ 3 & -4 & -1 & 2 \end{array}\right]$$

Here are the individual computations for this operation.

$$\begin{aligned} \text{Column 1} &: \; 1 - 6(3) = -17 \\ \text{Column 2} &: \; -6 - 6(-4) = 18 \\ \text{Column 3} &: \; 2 - 6(-1) = 8 \\ \text{Column 4} &: \; 0 - 6(2) = -12 \end{aligned}$$

3. For the following augmented matrix perform the indicated elementary row operations.

$$\left[\begin{array}{ccc|c} 10 & -1 & -5 & 1 \\ 4 & 0 & 7 & -1 \\ 0 & 7 & -2 & 3 \end{array}\right]$$

(a) $-9R_3$ **(b)** $R_1 \leftrightarrow R_2$ **(c)** $R_3 - R_1 \to R_3$

Solutions

(a) $-9R_3$

Solution

This operation is telling us to multiply all the entries in Row 3 of the augmented matrix by -9 so let's do that.

$$\left[\begin{array}{ccc|c} 10 & -1 & -5 & 1 \\ 4 & 0 & 7 & -1 \\ 0 & 7 & -2 & 3 \end{array}\right] \xrightarrow{-9R_3} \left[\begin{array}{ccc|c} 10 & -1 & -5 & 1 \\ 4 & 0 & 7 & -1 \\ 0 & -63 & 18 & -27 \end{array}\right]$$

(b) $R_1 \leftrightarrow R_2$

Solution

This operation is telling us to interchange Row 1 and Row 2 of the augmented matrix. Here is that work.

$$\left[\begin{array}{ccc|c} 10 & -1 & -5 & 1 \\ 4 & 0 & 7 & -1 \\ 0 & 7 & -2 & 3 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 4 & 0 & 7 & -1 \\ 10 & -1 & -5 & 1 \\ 0 & 7 & -2 & 3 \end{array}\right]$$

(c) $R_3 - R_1 \to R_3$

Solution

For this operation we are going to replace Row 3 with the results of taking the original entries from Row 3 and subtract from them the entries in Row 1.

$$\left[\begin{array}{rrr|r} 10 & -1 & -5 & 1 \\ 4 & 0 & 7 & -1 \\ 0 & 7 & -2 & 3 \end{array}\right] \begin{array}{c} R_3 - R_1 \to R_3 \\ \to \end{array} \left[\begin{array}{rrr|r} 10 & -1 & -5 & 1 \\ 4 & 0 & 7 & -1 \\ -10 & 8 & 3 & 2 \end{array}\right]$$

Here are the individual computations for this operation.

$$\begin{aligned} \text{Column 1} &: \ 0 - (10) = -10 \\ \text{Column 2} &: \ 7 - (-1) = 8 \\ \text{Column 3} &: \ -2 - (-5) = 3 \\ \text{Column 4} &: \ 3 - (1) = 2 \end{aligned}$$

7.4 More on the Augmented MatrixII

1. For the following system of equations convert the system into an augmented matrix and use the augmented matrix techniques to determine the solution to the system or to determine if the system is inconsistent or dependent.

$$\begin{aligned} x - 7y &= -11 \\ 5x + 2y &= -18 \end{aligned}$$

Step 1

The first step is to write down the augmented matrix for the system of equations.

$$\left[\begin{array}{cc|c} 1 & -7 & -11 \\ 5 & 2 & -18 \end{array}\right]$$

Step 2

We need to make the number in the upper left corner a one. In this case it already is and so there really isn't anything to do in this step for this particular problem.

Step 3

Next, we need to convert the 5 below the 1 into a zero and we can do that with the following elementary row operation.

$$\left[\begin{array}{cc|c} 1 & -7 & -11 \\ 5 & 2 & -18 \end{array}\right] \quad \underrightarrow{R_2 - 5R_1 \to R_2} \quad \left[\begin{array}{cc|c} 1 & -7 & -11 \\ 0 & 37 & 37 \end{array}\right]$$

Step 4

The next step is to turn the number at the bottom of the second column (37 in this case) into a one. The following elementary row operation will do that for us.

$$\left[\begin{array}{cc|c} 1 & -7 & -11 \\ 0 & 37 & 37 \end{array}\right] \quad \underrightarrow{\tfrac{1}{37}R_2} \quad \left[\begin{array}{cc|c} 1 & -7 & -11 \\ 0 & 1 & 1 \end{array}\right]$$

Step 5

Finally, we need to convert the number above the one we got in Step 4 into a zero. To do that we can use the following elementary row operation.

$$\left[\begin{array}{cc|c} 1 & -7 & -11 \\ 0 & 1 & 1 \end{array}\right] \quad \underset{\rightarrow}{R_1 + 7R_2 \to R_1} \quad \left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 1 \end{array}\right]$$

Step 6

From the final augmented matrix we found in Step 5 we get the solution to the system is : $\boxed{x = -4, \;\; y = 1}$.

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2. For the following system of equations convert the system into an augmented matrix and use the augmented matrix techniques to determine the solution to the system or to determine if the system is inconsistent or dependent.

$$\begin{aligned} 7x - 8y &= -12 \\ -4x + 2y &= 3 \end{aligned}$$

Step 1

The first step is to write down the augmented matrix for the system of equations.

$$\left[\begin{array}{cc|c} 7 & -8 & -12 \\ -4 & 2 & 3 \end{array}\right]$$

Step 2

We need to make the number in the upper left corner a one. There are several ways to do this. One way would be to use the elementary row operation $\frac{1}{7}R_1$. However, this would put fractions into the other two entries in the first row and it might be nice to avoid them.

So, instead let's do the following elementary row operation.

$$\left[\begin{array}{cc|c} 7 & -8 & -12 \\ -4 & 2 & 3 \end{array}\right] \quad \underset{\rightarrow}{R_1 + 2R_2 \to R_1} \quad \left[\begin{array}{cc|c} -1 & -4 & -6 \\ -4 & 2 & 3 \end{array}\right]$$

Now, this isn't quite what we want since the number in the upper left is a minus one and not a positive one. However, we can easily fix that by multiplying the first row by -1.

$$\left[\begin{array}{rr|r} -1 & -4 & -6 \\ -4 & 2 & 3 \end{array}\right] \quad \begin{array}{c} -R_1 \\ \rightarrow \end{array} \quad \left[\begin{array}{rr|r} 1 & 4 & 6 \\ -4 & 2 & 3 \end{array}\right]$$

Note that as this step has shown there are several different paths to do these problems. Some will result in "messier" intermediate steps, but the solution we get in the end will be the same regardless of the path we chose to follow in the solution process.

Step 3

Next, we need to convert the -4 below the 1 into a zero and we can do that with the following elementary row operation.

$$\left[\begin{array}{rr|r} 1 & 4 & 6 \\ -4 & 2 & 3 \end{array}\right] \quad \begin{array}{c} R_2 + 4R_1 \rightarrow R_2 \\ \rightarrow \end{array} \quad \left[\begin{array}{rr|r} 1 & 4 & 6 \\ 0 & 18 & 27 \end{array}\right]$$

Step 4

The next step is to turn the number at the bottom of the second column (18 in this case) into a one. The following elementary row operation will do that for us.

$$\left[\begin{array}{rr|r} 1 & 4 & 6 \\ 0 & 18 & 27 \end{array}\right] \quad \begin{array}{c} \frac{1}{18}R_2 \\ \rightarrow \end{array} \quad \left[\begin{array}{rr|r} 1 & 4 & 6 \\ 0 & 1 & \frac{27}{18} \end{array}\right] = \left[\begin{array}{rr|r} 1 & 4 & 6 \\ 0 & 1 & \frac{3}{2} \end{array}\right]$$

In the first step we chose to avoid the step that put fractions into the augmented matrix, but sometimes, as in this step, they can't be avoided.

Step 5

Finally, we need to convert the number above the one we got in Step 4 into a zero. To do that we can use the following elementary row operation.

$$\left[\begin{array}{rr|r} 1 & 4 & 6 \\ 0 & 1 & \frac{3}{2} \end{array}\right] \quad \begin{array}{c} R_1 - 4R_2 \rightarrow R_1 \\ \rightarrow \end{array} \quad \left[\begin{array}{rr|r} 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} \end{array}\right]$$

Step 6

From the final augmented matrix we found in Step 5 we get the solution to the system is :

$$\boxed{x = 0,\ y = \frac{3}{2}}.$$

---

3. For the following system of equations convert the system into an augmented matrix and use the augmented matrix techniques to determine the solution to the system or to determine if the system is inconsistent or dependent.

$$\begin{aligned} 3x + 9y &= -6 \\ -4x - 12y &= 8 \end{aligned}$$

Step 1

The first step is to write down the augmented matrix for the system of equations.

$$\left[\begin{array}{rr|r} 3 & 9 & -6 \\ -4 & -12 & 8 \end{array}\right]$$

Step 2

We need to make the number in the upper left corner a one. In this case we can quickly do that by dividing the top row by 3.

$$\left[\begin{array}{rr|r} 3 & 9 & -6 \\ -4 & -12 & 8 \end{array}\right] \quad \begin{array}{c} \frac{1}{3}R_1 \\ \to \end{array} \quad \left[\begin{array}{rr|r} 1 & 3 & -2 \\ -4 & -12 & 8 \end{array}\right]$$

Step 3

Next, we need to convert the -4 below the 1 into a zero and we can do that with the following elementary row operation.

$$\left[\begin{array}{rr|r} 1 & 3 & -2 \\ -4 & -12 & 8 \end{array}\right] \quad \begin{array}{c} R_2 + 4R_1 \to R_2 \\ \to \end{array} \quad \left[\begin{array}{rr|r} 1 & 3 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

Step 4

The minute we see the bottom row of all zeroes we know that the system if **dependent**. We can convert the top row into an equation and solve for x as follows,

$$x + 3y = -2 \qquad \rightarrow \qquad x = -3y - 2$$

From this we can write the solution as,

$$\begin{aligned} x &= -3t - 2 \\ y &= t \end{aligned} \qquad t \text{ is any number}$$

---

4. For the following system of equations convert the system into an augmented matrix and use the augmented matrix techniques to determine the solution to the system or to determine if the system is inconsistent or dependent.

$$\begin{aligned} 6x - 5y &= 8 \\ -12x + 2y &= 0 \end{aligned}$$

Step 1

The first step is to write down the augmented matrix for the system of equations.

$$\left[\begin{array}{rr|r} 6 & -5 & 8 \\ -12 & 2 & 0 \end{array}\right]$$

Step 2

We need to make the number in the upper left corner a one. There are several ways to do this. One way would be to use the elementary row operation $\frac{1}{6}R_1$. However, this would put fractions into the other two entries in the first row.

We're not going to be able to avoid fractions after this step and the above idea would do what we need but it would lead to two fractions. Note however that if we interchange the two rows we get,

$$\left[\begin{array}{rr|r} 6 & -5 & 8 \\ -12 & 2 & 0 \end{array}\right] \quad \begin{array}{c} R_1 \leftrightarrow R_2 \\ \rightarrow \end{array} \quad \left[\begin{array}{rr|r} -12 & 2 & 0 \\ 6 & -5 & 8 \end{array}\right]$$

We could now do the elementary row operation $-\frac{1}{12}R_1$ and we'll only end up with one fraction in the first row instead of two so let's do that.

$$\left[\begin{array}{rr|r} -12 & 2 & 0 \\ 6 & -5 & 8 \end{array}\right] \quad \underset{\rightarrow}{-\tfrac{1}{12}R_1} \quad \left[\begin{array}{rr|r} 1 & -\frac{1}{6} & 0 \\ 6 & -5 & 8 \end{array}\right]$$

Note that as this step has shown there are several different paths to do these problems. Some will result in "messier" intermediate steps, but the solution we get in the end will be the same regardless of the path we chose to follow in the solution process.

Step 3

Next, we need to convert the 6 below the 1 into a zero and we can do that with the following elementary row operation.

$$\left[\begin{array}{rr|r} 1 & -\frac{1}{6} & 0 \\ 6 & -5 & 8 \end{array}\right] \quad \underset{\rightarrow}{R_2 - 6R_1 \rightarrow R_2} \quad \left[\begin{array}{rr|r} 1 & -\frac{1}{6} & 0 \\ 0 & -4 & 8 \end{array}\right]$$

Step 4

The next step is to turn the number at the bottom of the second column (-4 in this case) into a one. The following elementary row operation will do that for us.

$$\left[\begin{array}{rr|r} 1 & -\frac{1}{6} & 0 \\ 0 & -4 & 8 \end{array}\right] \quad \underset{\rightarrow}{-\tfrac{1}{4}R_2} \quad \left[\begin{array}{rr|r} 1 & -\frac{1}{6} & 0 \\ 0 & 1 & -2 \end{array}\right]$$

Step 5

Finally, we need to convert the number above the one we got in Step 4 into a zero. To do that we can use the following elementary row operation.

$$\left[\begin{array}{rr|r} 1 & -\frac{1}{6} & 0 \\ 0 & 1 & -2 \end{array}\right] \quad \underset{\rightarrow}{R_1 + \tfrac{1}{6}R_2 \rightarrow R_1} \quad \left[\begin{array}{rr|r} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -2 \end{array}\right]$$

Step 6

From the final augmented matrix we found in Step 5 we get the solution to the system is :

$$\boxed{x = -\frac{1}{3},\ y = -2}.$$

5. For the following system of equations convert the system into an augmented matrix and use the augmented matrix techniques to determine the solution to the system or to determine if the system is inconsistent or dependent.

$$\begin{aligned} 5x - 25y &= 3 \\ -2x + 10y &= 2 \end{aligned}$$

Step 1

The first step is to write down the augmented matrix for the system of equations.

$$\left[\begin{array}{rr|r} 5 & -25 & 3 \\ -2 & 10 & 2 \end{array}\right]$$

Step 2

We need to make the number in the upper left corner a one. In this case we can do this with the following elementary row operation.

$$\left[\begin{array}{rr|r} 5 & -25 & 3 \\ -2 & 10 & 2 \end{array}\right] \quad \begin{array}{c} R_1 + \frac{5}{2}R_2 \to R_1 \\ \to \end{array} \quad \left[\begin{array}{rr|r} 0 & 0 & 8 \\ -2 & 10 & 2 \end{array}\right]$$

Step 3

Okay let's step back for a second and convert the first row back to an equation. Doing this gives,

$$0 = 8$$

That is clearly not true and we've done all our work correctly and so this system is **inconsistent** and there is **no solution** to the system.

6. For the following system of equations convert the system into an augmented matrix and use the augmented matrix techniques to determine the solution to the system or to determine if the system

is inconsistent or dependent.

$$\begin{aligned} 2x + 3y &= 20 \\ 7x + 2y &= 53 \end{aligned}$$

Step 1

The first step is to write down the augmented matrix for the system of equations.

$$\left[\begin{array}{cc|c} 2 & 3 & 20 \\ 7 & 2 & 53 \end{array}\right]$$

Step 2

We need to make the number in the upper left corner a one. There are several ways to do this. One way would be to use the elementary row operation $\frac{1}{2}R_1$. However, this would put fractions into the other two entries in the first row and it might be nice to avoid them.

While this may seem to not be of any use let's take a look at the following elementary row operation.

$$\left[\begin{array}{cc|c} 2 & 3 & 20 \\ 7 & 2 & 53 \end{array}\right] \quad \begin{array}{c} R_2 - 3R_1 \to R_2 \\ \to \end{array} \quad \left[\begin{array}{cc|c} 2 & 3 & 20 \\ 1 & -7 & -7 \end{array}\right]$$

This operation worked on the second row instead of the first row that we need to work on. Note however, that we did put a 1 in the lower number of the first column. We need a 1 in the upper number of the first column and we can do that now simply by switching rows as follows,

$$\left[\begin{array}{cc|c} 2 & 3 & 20 \\ 1 & -7 & -7 \end{array}\right] \quad \begin{array}{c} R_1 \leftrightarrow R_2 \\ \to \end{array} \quad \left[\begin{array}{cc|c} 1 & -7 & -7 \\ 2 & 3 & 20 \end{array}\right]$$

Note that as this step has shown there are several different paths to do these problems. Some will result in "messier" intermediate steps, but the solution we get in the end will be the same regardless of the path we chose to follow in the solution process.

Step 3

Next, we need to convert the 2 below the 1 into a zero and we can do that with the following elementary row operation.

$$\left[\begin{array}{cc|c} 1 & -7 & -7 \\ 2 & 3 & 20 \end{array}\right] \quad \begin{array}{c} R_2 - 2R_1 \to R_2 \\ \to \end{array} \quad \left[\begin{array}{cc|c} 1 & -7 & -7 \\ 0 & 17 & 34 \end{array}\right]$$

Step 4

The next step is to turn the number at the bottom of the second column (17 in this case) into a one. The following elementary row operation will do that for us.

$$\left[\begin{array}{cc|c} 1 & -7 & -7 \\ 0 & 17 & 34 \end{array}\right] \quad \begin{array}{c} \frac{1}{17}R_2 \\ \rightarrow \end{array} \quad \left[\begin{array}{cc|c} 1 & -7 & -7 \\ 0 & 1 & 2 \end{array}\right]$$

Step 5

Finally, we need to convert the number above the one we got in Step 4 into a zero. To do that we can use the following elementary row operation.

$$\left[\begin{array}{cc|c} 1 & -7 & -7 \\ 0 & 1 & 2 \end{array}\right] \quad \begin{array}{c} R_2 + 7R_1 \rightarrow R_2 \\ \rightarrow \end{array} \quad \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 2 \end{array}\right]$$

Step 6

From the final augmented matrix we found in Step 5 we get the solution to the system is : $\boxed{x = 7,\ y = 2}$.

7. For the following system of equations convert the system into an augmented matrix and use the augmented matrix techniques to determine the solution to the system or to determine if the system is inconsistent or dependent.

$$\begin{aligned} 2x + 5y + 2z &= -38 \\ 3x - 2y + 4z &= 17 \\ -6x + y - 7z &= -12 \end{aligned}$$

Step 1

The first step is to write down the augmented matrix for the system of equations.

$$\left[\begin{array}{rrr|r} 2 & 5 & 2 & -38 \\ 3 & -2 & 4 & 17 \\ -6 & 1 & -7 & -12 \end{array}\right]$$

Step 2

We need to make the number in the upper left corner a one. Much like with the previous problems (*i.e.* solving systems with two variables) we can quickly do it with the elementary row operation $\frac{1}{2}R_1$ but that will put fractions into the augmented matrix and they would probably be around for quite a few steps and it would be really nice to avoid them for as long as possible when the augmented matrix starts getting this size.

So, let's start with the following elementary row operation.

$$\left[\begin{array}{rrr|r} 2 & 5 & 2 & -38 \\ 3 & -2 & 4 & 17 \\ -6 & 1 & -7 & -12 \end{array}\right] \quad \begin{array}{c} R_1 - R_2 \to R_1 \\ \to \end{array} \quad \left[\begin{array}{rrr|r} -1 & 7 & -2 & -55 \\ 3 & -2 & 4 & 17 \\ -6 & 1 & -7 & -12 \end{array}\right]$$

With this operation we got a negative one in the spot where we needed a plus one, but we can easily fix that with the next elementary row operation.

$$\left[\begin{array}{rrr|r} -1 & 7 & -2 & -55 \\ 3 & -2 & 4 & 17 \\ -6 & 1 & -7 & -12 \end{array}\right] \quad \begin{array}{c} -R_1 \\ \to \end{array} \quad \left[\begin{array}{rrrr} 1 & -7 & 2 & 55 \\ 3 & -2 & 4 & 17 \\ -6 & 1 & -7 & -12 \end{array}\right]$$

Now, a quick note before we really jump into the rest of this problem. Using augmented matrices to solve systems with three variables can be a very tedious process and there are a great number of possible paths to take in the solution process so your solution may well vary from this solution depending on the path you took. The final answers however will the same regardless of the path we take provided we did all the arithmetic correctly.

Step 3

Next, we need to convert the 3 and the -6 below the 1 in the first column into zeroes and we can do that with the following elementary row operations.

$$\left[\begin{array}{rrr|r} 1 & -7 & 2 & 55 \\ 3 & -2 & 4 & 17 \\ -6 & 1 & -7 & -12 \end{array}\right] \quad \begin{array}{r} R_2 - 3R_1 \to R_2 \\ R_3 + 6R_1 \to R_3 \\ \to \end{array} \quad \left[\begin{array}{rrrr} 1 & -7 & 2 & 55 \\ 0 & 19 & -2 & -148 \\ 0 & -41 & 5 & 318 \end{array}\right]$$

Step 4

We now need to turn the 19 in the second row into a one and it seems like the only easy way to do that is the following elementary row operation.

$$\left[\begin{array}{rrr|r} 1 & -7 & 2 & 55 \\ 0 & 19 & -2 & -148 \\ 0 & -41 & 5 & 318 \end{array}\right] \quad \begin{array}{c} \frac{1}{19}R_2 \\ \to \end{array} \quad \left[\begin{array}{rrr|r} 1 & -7 & 2 & 55 \\ 0 & 1 & -\frac{2}{19} & -\frac{148}{19} \\ 0 & -41 & 5 & 318 \end{array}\right]$$

In the first step we chose to avoid the step that put fractions into the augmented matrix, but sometimes, as in this step, they can't be avoided. With augmented matrices for systems with three variables fractions will almost inevitably show up and they will often be "messy" when they do.

This is just something we'll need to deal with when solving these systems. We try to avoid them for as long as possible but except it when they show up and continue with the solution process.

Step 5

Next, we need to turn the -41 in the third row into a zero. The following elementary row operation will do that for us.

$$\left[\begin{array}{rrr|r} 1 & -7 & 2 & 55 \\ 0 & 1 & -\frac{2}{19} & -\frac{148}{19} \\ 0 & -41 & 5 & 318 \end{array}\right] \quad \begin{array}{c} R_3 + 41R_2 \to R_3 \\ \to \end{array} \quad \left[\begin{array}{rrr|r} 1 & -7 & 2 & 55 \\ 0 & 1 & -\frac{2}{19} & -\frac{148}{19} \\ 0 & 0 & \frac{13}{19} & -\frac{26}{19} \end{array}\right]$$

Again, we had to put more fraction into the augmented matrix. This is just a fact of life with these types of problems. However, as we'll see in the next step they do often disappear as well.

Step 6

Okay, we need to turn the $\frac{13}{19}$ in the third row into a one and we can do that as follows,

$$\left[\begin{array}{rrr|r} 1 & -7 & 2 & 55 \\ 0 & 1 & -\frac{2}{19} & -\frac{148}{19} \\ 0 & 0 & \frac{13}{19} & -\frac{26}{19} \end{array}\right] \quad \underrightarrow{\frac{19}{13}R_3} \quad \left[\begin{array}{rrr|r} 1 & -7 & 2 & 55 \\ 0 & 1 & -\frac{2}{19} & -\frac{148}{19} \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Step 7

Next, we need to turn the $-\frac{2}{19}$ and the 2 in the third column into zeroes. The following elementary row operations will do that for us.

$$\left[\begin{array}{rrr|r} 1 & -7 & 2 & 55 \\ 0 & 1 & -\frac{2}{19} & -\frac{148}{19} \\ 0 & 0 & 1 & -2 \end{array}\right] \quad \begin{array}{c} R_1 - 2R_3 \to R_1 \\ R_2 + \frac{2}{19}R_3 \to R_2 \\ \to \end{array} \quad \left[\begin{array}{rrr|r} 1 & -7 & 0 & 59 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Note that the fractions are now completely gone! This won't always happen but it also will happen fairly regularly that fractions get introduced in intermediate steps and then go away in later steps.

Step 8

For the final operation we need to turn the -7 in the second column into a zero and we can do that as follows,

$$\left[\begin{array}{rrr|r} 1 & -7 & 0 & 59 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & -2 \end{array}\right] \quad \underrightarrow{R_1 + 7R_2 \to R_1} \quad \left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Step 9

From the final augmented matrix we found in Step 8 we get the solution to the system is : $\boxed{x = 3,\ y = -8,\ z = -2}$.

8. For the following system of equations convert the system into an augmented matrix and use the augmented matrix techniques to determine the solution to the system or to determine if the system

is inconsistent or dependent.

$$\begin{aligned} 3x - 9z &= 33 \\ 7x - 4y - z &= -15 \\ 4x + 6y + 5z &= -6 \end{aligned}$$

Step 1

The first step is to write down the augmented matrix for the system of equations.

$$\left[\begin{array}{ccc|c} 3 & 0 & -9 & 33 \\ 7 & -4 & -1 & -15 \\ 4 & 6 & 5 & -6 \end{array}\right]$$

Note the zero in the second column of the first row. Recall that the second column corresponds to the coefficients of the y's in each equation and because there is no y in the first equation that coefficient must be zero.

Step 2

We need to make the number in the upper left corner a one. We can easily do that with the following elementary row operation.

$$\left[\begin{array}{ccc|c} 3 & 0 & -9 & 33 \\ 7 & -4 & -1 & -15 \\ 4 & 6 & 5 & -6 \end{array}\right] \quad \begin{array}{c} \frac{1}{3}R_1 \\ \rightarrow \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & -3 & 11 \\ 7 & -4 & -1 & -15 \\ 4 & 6 & 5 & -6 \end{array}\right]$$

Step 3

Next, we need to convert the 7 and the 4 below the 1 in the first column into zeroes and we can do that with the following elementary row operations.

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 11 \\ 7 & -4 & -1 & -15 \\ 4 & 6 & 5 & -6 \end{array}\right] \quad \begin{array}{c} R_2 - 7R_1 \rightarrow R_2 \\ R_3 - 4R_1 \rightarrow R_3 \\ \rightarrow \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & -3 & 11 \\ 0 & -4 & 20 & -92 \\ 0 & 6 & 17 & -50 \end{array}\right]$$

Step 4

We now need to turn the -4 in the second row into a one and that can be done with the following elementary row operation.

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 11 \\ 0 & -4 & 20 & -92 \\ 0 & 6 & 17 & -50 \end{array}\right] \quad \begin{array}{c} -\frac{1}{4}R_2 \\ \rightarrow \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & -3 & 11 \\ 0 & 1 & -5 & 23 \\ 0 & 6 & 17 & -50 \end{array}\right]$$

Step 5

Next, we need to turn the 6 in the third row into a zero. The following elementary row operation will do that for us.

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 11 \\ 0 & 1 & -5 & 23 \\ 0 & 6 & 17 & -50 \end{array}\right] \quad \begin{array}{c} R_3 - 6R_2 \rightarrow R_3 \\ \rightarrow \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & -3 & 11 \\ 0 & 1 & -5 & 23 \\ 0 & 0 & 47 & -188 \end{array}\right]$$

Step 6

Okay, we need to turn the 47 in the third row into a one and we can do that as follows,

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 11 \\ 0 & 1 & -5 & 23 \\ 0 & 0 & 47 & -188 \end{array}\right] \quad \begin{array}{c} \frac{1}{47}R_3 \\ \rightarrow \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & -3 & 11 \\ 0 & 1 & -5 & 23 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

Step 7

Next, we need to turn the -5 and the -3 in the third column into zeroes. The following elementary row operations will do that for us.

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 11 \\ 0 & 1 & -5 & 23 \\ 0 & 0 & 1 & -4 \end{array}\right] \quad \begin{array}{c} R_1 + 3R_3 \rightarrow R_1 \\ R_2 + 5R_3 \rightarrow R_2 \\ \rightarrow \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

Step 8

Normally we would have another step to do. We would need to turn the number in the first row and second column into a zero. However, in this case there is already a zero there and so there is no work to do in this step.

The final form of the augmented matrix is then,

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

As this step has shown we occasionally will get a number "for free". In other words, the work we put into an intermediate step will give us not only the number we were looking for in that step but will also put in a number that we need in a later step. Or, as in this case, the number we needed was actually there from the start.

Step 9

From the final augmented matrix we found in Step 8 we get the solution to the system is :

$\boxed{x = -1,\ y = 3,\ z = -4}$.

7.5 NonlinearSystems

1. Find the solution to the following system of equation.

$$\begin{aligned} y &= x^2 + 6x - 8 \\ y &= 4x + 7 \end{aligned}$$

Step 1

Before we get too far into the solution we first should mention that there is no one correct solution path to these. Many of these types of problems will have multiple paths that we can take to find the solution. However, regardless of the path we take the solution to the system will be the same.

Okay on to the problem. In this case we can notice that both of the equations are in the form "y =". This means that we can "substitute" y from one of the equations into the other. In these kinds of problems this is often called "setting the equations equal".

So, setting the equations equal gives,

$$x^2 + 6x - 8 = 4x + 7$$

Step 2

Now, this is just a quadratic equation and by this point we should be able to solve that so here is the solution work for the quadratic.

$$\begin{aligned} x^2 + 6x - 8 &= 4x + 7 \\ x^2 + 2x - 15 &= 0 \\ (x-3)(x+5) &= 0 \qquad \to \qquad x = -5, \quad x = 3 \end{aligned}$$

Step 3

We now have two values of x and so all we need to do is plug into either of the original equations (the line would be easier) to determine the corresponding values of y for each x.

$$\begin{aligned} x = -5 &: \quad y = 4(-5) + 7 = -13 \qquad \Rightarrow \qquad (-5, -13) \\ x = 3 &: \quad y = 4(3) + 7 = 19 \qquad \Rightarrow \qquad (3, 19) \end{aligned}$$

So, for this system of equations we have two solutions : $\boxed{(-5, -13)}$ and $\boxed{(3, 19)}$
.

2. Find the solution to the following system of equation.

$$y = 1 - 3x$$
$$\frac{x^2}{4} + y^2 = 1$$

Step 1

Before we get too far into the solution we first should mention that there is no one correct solution path to these. Many of these types of problems will have multiple paths that we can take to find the solution. However, regardless of the path we take the solution to the system will be the same.

Okay on to the problem. In this case the first equation is in the form "y =" and so we can just plug this directly into the second equation. Doing this gives,

$$\frac{x^2}{4} + (1 - 3x)^2 = 1$$

Be careful with the parenthesis when plugging the first equation in. We had y^2 and so we need to make sure and square the whole y form the first equation when we plugged that in. In other words, we need the parenthesis in there to make sure we deal with the exponent properly.

Step 2

Now, this is just a quadratic equation (which admittedly needs some simplification) and by this point we should be able to solve that so here is the solution work for the quadratic.

$$\begin{aligned} \frac{x^2}{4} + (1 - 3x)^2 &= 1 \\ \frac{x^2}{4} + 1 - 6x + 9x^2 - 1 &= 0 \\ \frac{37}{4}x^2 - 6x &= 0 \\ x\left(\frac{37}{4}x - 6\right) &= 0 \qquad \rightarrow \qquad x = 0, \quad \frac{37}{4}x - 6 = 0 \end{aligned}$$

From this we see we have two values of x for our solution : $x = 0$ and $x = \frac{24}{37}$.

Step 3

We now have two values of x and so all we need to do is plug into either of the original equations (the line would be easier) to determine the corresponding values of y for each x.

$$x = 0 \quad : \quad y = 1 - 3(0) = 1 \qquad \Rightarrow \qquad (0,1)$$
$$x = \frac{24}{37} \quad : \quad y = 1 - 3\left(\frac{24}{37}\right) = -\frac{35}{37} \qquad \Rightarrow \qquad \left(\frac{24}{37}, -\frac{35}{37}\right)$$

Note that with this system we have also run into a potential problem. We found corresponding y's by plugging our x's into the line. What if we had plugged them into the ellipse?

Let's use $x = 0$ as an example since it will be a little easier to do the work for. Plugging this into the equation for the ellipse gives,

$$\frac{(0)^2}{4} + y^2 = 1 \quad \rightarrow \quad y^2 = 1 \quad \rightarrow \quad y = \pm 1$$

This implies that there should be two solutions corresponding to $x = 0$ and not the single solution we found above! So, which is correct? Well recall that whatever solution we get must satisfy ***both*** equations and only one of those values of y will satisfy the line when using $x = 0$. Therefore, the only solution is the one we got from the line. We would have a similar issue with the second value of x.

The problem arose when we plugged the line into the ellipse and squared it. The process of squaring it introduced potentially "bad" solutions. We saw similar issues when we solved equations with radicals several chapters back and the problem arose there for the same reason, we squared something.

The nice thing about these problems however is that if we use the equation we plugged in (the line in this case) to find the second values we don't need to worry about the "bad" solutions since they only arise from the equation that we plugged into (the ellipse in this case).

This in fact is the real reason we used the line to find the corresponding y, although it was also the easier of the two equations to use.

So, for this system of equations we have two solutions : $\boxed{(0,1)}$ and $\boxed{\left(\frac{24}{37}, -\frac{35}{37}\right)}$

.

3. Find the solution to the following system of equation.

$$xy = 4$$
$$\frac{x^2}{4} + \frac{y^2}{25} = 1$$

Step 1

Before we get too far into the solution we first should mention that there is no one correct solution path to these. Many of these types of problems will have multiple paths that we can take to find the solution. However, regardless of the path we take the solution to the system will be the same.

Okay on to the problem. In this case we can solve the first equation for either x or y and plug this into the second equation. For no reason other than we had equations in x for the first two practice problems for this section we'll solve the first equation for x and plug this into the second equation. The result will be an equation involving only y's.

Here is that work.

$$x = \frac{4}{y} \qquad \rightarrow \qquad \begin{aligned} \frac{\left(\frac{4}{y}\right)^2}{4} + \frac{y^2}{25} &= 1 \\ \frac{\frac{16}{y^2}}{4} + \frac{y^2}{25} &= 1 \\ \frac{4}{y^2} + \frac{y^2}{25} &= 1 \end{aligned}$$

Step 2

Now, let's multiply both sides of this by $25y^2$ to clear denominators.

$$\begin{aligned} 100 + y^4 &= 25y^2 \\ y^4 - 25y^2 + 100 &= 0 \end{aligned}$$

Step 3

This is quadratic in form so we can define $u = y^2$ (and so $u^2 = \left(y^2\right)^2 = y^4$). Using this substitution the equation becomes,

$$\begin{aligned} u^2 - 25u + 100 &= 0 \\ (u-5)(u-20) &= 0 \qquad \rightarrow \qquad u = 5, \quad u = 20 \end{aligned}$$

Step 4

So, we got two values of u and each of these correspond to the following equation in terms of y (*i.e.* using the substitution above).

$$\begin{aligned} u &= 5 \;: \quad y^2 = 5 & &\rightarrow & y &= \pm\sqrt{5} \\ u &= 20 : \quad y^2 = 20 & &\rightarrow & y &= \pm\sqrt{20} = \pm 2\sqrt{5} \end{aligned}$$

Step 5

We have four values of y that we need to find corresponding values of x for. We'll plug these into the first equation (much easier to plug these into that equation).

$$\begin{aligned} y &= \sqrt{5} \;: \quad x = \frac{4}{\sqrt{5}} & &\Rightarrow & &\left(\frac{4}{\sqrt{5}}, \sqrt{5}\right) \\ y &= -\sqrt{5} \;: \quad x = \frac{4}{-\sqrt{5}} & &\Rightarrow & &\left(-\frac{4}{\sqrt{5}}, -\sqrt{5}\right) \\ y &= 2\sqrt{5} \;: \quad x = \frac{4}{2\sqrt{5}} & &\Rightarrow & &\left(\frac{2}{\sqrt{5}}, 2\sqrt{5}\right) \\ y &= -2\sqrt{5} \;: \quad x = \frac{4}{-2\sqrt{5}} & &\Rightarrow & &\left(-\frac{2}{\sqrt{5}}, -2\sqrt{5}\right) \end{aligned}$$

So, for this system of equations we have the four solutions listed above.

4. Find the solution to the following system of equation.

$$\begin{aligned} y &= 1 - 2x^2 \\ x^2 - \frac{y^2}{9} &= 1 \end{aligned}$$

Step 1

Before we get too far into the solution we first should mention that there is no one correct solution path to these. Many of these types of problems will have multiple paths that we can take to find the solution. However, regardless of the path we take the solution to the system will be the same.

Okay on to the problem. In this case the first equation is in the form "y =" and so we can just plug this directly into the second equation. Doing this gives,

$$x^2 - \frac{\left(1 - 2x^2\right)^2}{9} = 1$$

Be careful with the parenthesis when plugging the first equation in. We had y^2 and so we need to make sure and square the whole y form the first equation when we plugged that in. In other words, we need the parenthesis in there to make sure we deal with the exponent properly.

Step 2

Now, this is just a quadratic equation (which admittedly needs some simplification) and by this point we should be able to solve that so here is the solution work for the quadratic.

$$\begin{aligned} x^2 - \frac{\left(1-2x^2\right)^2}{9} &= 1 \\ x^2 - \frac{1}{9}\left(1-4x^2+4x^4\right) - 1 &= 0 \\ -\frac{4}{9}x^4 + \frac{13}{9}x^2 - \frac{10}{9} &= 0 \\ 4x^4 - 13x^2 + 10 &= 0 \end{aligned}$$

In the last step we multiplied by -9 to clear out the denominators and to eliminate the minus sign on the x^4 term.

Step 3

This is quadratic in form so we can define $u = x^2$ (and so $u^2 = \left(x^2\right)^2 = x^4$). Using this substitution the equation becomes,

$$\begin{aligned} 4u^2 - 13u + 10 &= 0 \\ (4u-5)(u-2) &= 0 \qquad \to \qquad u = \frac{5}{4}, \quad u = 2 \end{aligned}$$

Step 4

So, we got two values of u and each of these correspond to the following equation in terms of y (*i.e.* using the substitution above).

$$\begin{aligned} u &= \frac{5}{4} \quad : \quad x^2 = \frac{5}{4} \qquad \to \qquad x = \frac{\pm\sqrt{5}}{2} \\ u &= 2 \quad : \quad x^2 = 2 \qquad \to \qquad x = \pm\sqrt{2} \end{aligned}$$

Step 5

We have four values of x that we need to find corresponding values of y for. We'll plug these into the first equation (much easier to plug these into that equation).

$$x = \frac{\sqrt{5}}{2} \quad : \quad y = 1 - 2\left(\frac{\sqrt{5}}{2}\right)^2 = -\frac{3}{2} \qquad \Rightarrow \qquad \left(\frac{\sqrt{5}}{2}, -\frac{3}{2}\right)$$
$$x = -\frac{\sqrt{5}}{2} \quad : \quad y = 1 - 2\left(-\frac{\sqrt{5}}{2}\right)^2 = -\frac{3}{2} \qquad \Rightarrow \qquad \left(-\frac{\sqrt{5}}{2}, -\frac{3}{2}\right)$$
$$x = \sqrt{2} \quad : \quad y = 1 - 2\left(\sqrt{2}\right)^2 = -3 \qquad \Rightarrow \qquad \left(\sqrt{2}, -3\right)$$
$$x = -\sqrt{2} \quad : \quad y = 1 - 2\left(-\sqrt{2}\right)^2 = -3 \qquad \Rightarrow \qquad \left(-\sqrt{2}, -3\right)$$

So, for this system of equations we have the four solutions listed above.

Note that with this system we have also run into a potential problem. We found corresponding y's by plugging our x's into the parabola. What if we had plugged them into the hyperbola?

Let's use $x = \sqrt{2}$ as an example since it will be a little easier to do the work for. Plugging this into the equation for the hyperbola gives,

$$\left(\sqrt{2}\right)^2 - \frac{y^2}{9} = 1 \quad \rightarrow \quad 2 - \frac{y^2}{9} = 1 \quad \rightarrow \quad y^2 = 9 \quad \rightarrow \quad y = \pm 3$$

This implies that there should be two solutions corresponding to $x = \sqrt{2}$ and not the single solution we found above! So, which is correct? Well recall that whatever solution we get must satisfy ***both*** equations and only one of those values of y will satisfy the parabola when using $x = \sqrt{2}$. Therefore, the only solution is the one we got from the parabola. We would have a similar issue with the other values of x.

The problem arose when we plugged the parabola into the hyperbola and squared it. The process of squaring it introduced potentially "bad" solutions. We saw similar issues when we solved equations with radicals several chapters back and the problem arose there for the same reason, we squared something.

The nice thing about these problems however is that if we use the equation we plugged in (the parabola in this case) to find the second values we don't need to worry about the "bad" solutions since they only arise from the equation that we plugged into (the hyperbola in this case).

This in fact is the real reason we used the parabola to find the corresponding y, although it was also the easier of the two equations to use.

Index

Quite a few entries in this index are duplicated in several places throughout the index. For example **Vertex** is listed both separately and as a sub entry of **Parabolas**. The reason for this is that I tried to anticipate (probably badly at times....) just where in the index a reader would be looking for a particular topic. Again to use Work as an example. A reader might look for Vertex as a separate entry or, maybe, upon realizing it related to Parabolas look for it there.

So, assuming that I've anticipated properly, you should be able to find most entries in the table no matter where you look (provided it's actually in the table of course).