### Limits

**Definitions**

- **Precise Definition:** We say \( \lim_{x \to a} f(x) = L \) if for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that whenever \( 0 < |x - a| < \delta \) then \( |f(x) - L| < \varepsilon \).
- **"Working" Definition:** We say \( \lim_{x \to a} f(x) = L \) if we can make \( f(x) \) as close to \( L \) as we want by taking \( x \) large enough and positive.

**Limit at Infinity:** We say \( \lim_{x \to \infty} f(x) = L \) if we can make \( f(x) \) as close to \( L \) as we want by taking \( x \) large enough and positive.

There is a similar definition for \( \lim_{x \to -\infty} f(x) = L \) except we require \( x \) large and negative.

**Infinite Limit:** We say \( \lim_{x \to \infty} f(x) = \infty \) if we can make \( f(x) \) arbitrarily large (and positive) by taking \( x \) sufficiently close to \( a \) (on either side of \( a \)) without letting \( x = a \).

**Right hand limit:** \( \lim_{x \to a^+} f(x) = L \). This has the same definition as the limit except it requires \( x > a \).

**Left hand limit:** \( \lim_{x \to a^-} f(x) = L \). This has the same definition as the limit except it requires \( x < a \).

**Relationship between the limit and one-sided limits**

- \( \lim_{x \to a} f(x) = L \) if and only if \( \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L \).
- \( \lim_{x \to a} f(x) \) does not exist if \( \lim_{x \to a^+} f(x) \) or \( \lim_{x \to a^-} f(x) \) do not exist or do not equal \( L \).

**Properties**

1. \( \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \)
2. \( \lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) \)
3. \( \lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x) \)
4. \( \lim_{x \to a} \left[ \frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \) provided \( \lim_{x \to a} g(x) \neq 0 \)
5. \( \lim_{x \to a} [f(x)]^n = \left[ \lim_{x \to a} f(x) \right]^n \)
6. \( \lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} \)

**Basic Limit Evaluations at \( \pm \infty \)**

- \( \lim_{x \to \pm \infty} e^x = \infty \) and \( \lim_{x \to \pm \infty} e^{-x} = 0 \)
- \( \lim_{x \to \pm \infty} \ln(x) = \pm \infty \) and \( \lim_{x \to 0^+} \ln(x) = -\infty \)
- \( \lim_{x \to \pm \infty} x^n = \pm \infty \) for even \( n \)
- \( \lim_{x \to \pm \infty} x^n = -\infty \) for odd \( n \)
- \( \lim_{x \to \pm \infty} ax^n + \ldots + bx + c = \text{sgn}(a)\infty \)
- \( \lim_{x \to \pm \infty} ax^n + \ldots + bx + c = \text{sgn}(a)\text{sgn}(b)\infty \)
- \( \lim_{x \to \pm \infty} ax^n + \ldots + cx + d = -\infty \)

**Examples**

- \( \lim_{x \to \infty} \frac{1}{x} = 0 \)
- \( \lim_{x \to -\infty} \frac{1}{x} = 0 \)
- \( \lim_{x \to \infty} \frac{1}{x^2} = 0 \)
- \( \lim_{x \to -\infty} \frac{1}{x^2} = 0 \)

**Note:** \( \text{sgn}(a) = 1 \) if \( a > 0 \) and \( \text{sgn}(a) = -1 \) if \( a < 0 \).

### Calculus Cheat Sheet

**Continuous Functions**

- If \( f(x) \) is continuous at \( a \) then \( \lim_{x \to a} f(x) = f(a) \)
- \( \lim_{x \to \infty} \frac{x^2 + 4x - 12}{x^2 - 2x} = \lim_{x \to \infty} \frac{(x-2)(x+6)}{x(x-2)} = \lim_{x \to \infty} \frac{x+6}{x} = 1 \)

**Evaluation Techniques**

**L'Hospital's Rule**

- If \( \lim_{x \to a} f(x) = 0 \) or \( \lim_{x \to a} g(x) = 0 \) then \( \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \) if \( f'(x) \) and \( g'(x) \) exist.

**Polynomials at Infinity**

- If \( p(x) \) and \( q(x) \) are polynomials. To compute \( \lim_{x \to \infty} \frac{p(x)}{q(x)} \) factor largest power of \( x \) in \( q(x) \) out of both \( p(x) \) and \( q(x) \), then compute limit.

- \( \lim_{x \to \infty} \frac{3x^2 - 4}{5x^2 - 2x^2} = \lim_{x \to \infty} \frac{3x^2}{7x^2} = \frac{3}{7} \)

**Piecewise Function**

- \( \lim_{x \to a} g(x) \) where \( g(x) = \begin{cases} x^2 + 5 & \text{if } x < -2 \\ 1 - 3x & \text{if } x \geq -2 \end{cases} \)

Compute two one sided limits,

- \( \lim_{x \to -2} g(x) = \lim_{x \to -2^-} g(x) = 1 - 3(-2) = 7 \)
- \( \lim_{x \to -2} g(x) = \lim_{x \to -2^+} g(x) = (-2)^2 + 5 = 9 \)

One sided limits are different so \( \lim_{x \to -2} g(x) \) doesn’t exist. If the two one sided limits had been equal then \( \lim_{x \to a} g(x) \) would have existed and had the same value.

### Some Continuous Functions

Partial list of continuous functions and the values of \( x \) for which they are continuous.

1. Polynomials for all \( x \),
2. Rational function, except for \( x \)'s that give division by zero,
3. \( \sqrt{\ } \) (odd) for all \( x \),
4. \( \sqrt{\ } \) (even) for all \( x > 0 \),
5. \( e^x \) for all \( x \),
6. \( \ln(x) \) for \( x > 0 \),
7. \( \cos(x) \) and \( \sin(x) \) for all \( x \),
8. \( \tan(x) \) and \( \sec(x) \) provided \( x \neq \cdots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \cdots \),
9. \( \cot(x) \) and \( \csc(x) \) provided \( x \neq \cdots, -2\pi, -\pi, 0, \pi, 2\pi, \cdots \)

**Intermediate Value Theorem**

Suppose that \( f(x) \) is continuous on \([a, b]\) and let \( M \) be any number between \( f(a) \) and \( f(b) \). Then there exists a number \( c \) such that \( a < c < b \) and \( f(c) = M \).